NUMERICAL SOLUTION FOR A WAVE EQUATION WITH BOUNDARY DAMPING

M.Sc. Ardian NAÇO, Prof. Lulëzim HANELLI[*], M.Sc. Bendise HUTI

Departamenti i Matematikës, Universiteti Politeknik, Tiranë

SHQIPERI

E-mail: *lh9355@yahoo.co.uk*

ABSTRACT

Numerical solution is developed for a wave equation with boundary damping arising in the study of the physical phenomena of the oscillations that occur in overhead power transmission lines and other flexible structures. The physical model is that of a string which is fixed at one end and the other end is attached to a dashpot system, where the damping generated by the dashpot is small. The mathematical model is an initial-boundary value problem for a weekly nonlinear hyperbolic differential equation with non-classical boundary conditions.

The numerical method proposed is of the type of characteristics. It takes advantage of the special mesh generated by the characteristic curves of the equations to be solved and the specialty of the initial-boundary conditions involved. The method seems also to minimize the difficulties of complicated algebraic procedures and the word-length problems introduced by the classical method of characteristics.

PËRMBLEDHJE

Në këtë punim zgjidhet numerikisht një problem diferencial valor me shuarje kufitare, që ndeshet në studimin e fenomenit fizik të lëkundjeve në linjat e tensionit të lartë dhe në struktura të tjera fleksible. Modeli fizik është ai i një korde njëri skaj i së cilës qëndron i fiksuar, kurse skai tjetër lidhet me një sistem që gjeneron lëkundje, me koeficient shuarje të vogël. Modeli matematik është një problem i vlerës fillestare-kufitare për një ekuacion diferencial jolinear të dobët me kushte kufitare jo klasike.

Metoda numerike e propozuar është e tipit të karakteristikave. Ajo shfrytëzon rrjetin specific të nyjeve që gjenerojnë kurbat karakteristike të ekuacionit si dhe vecoritë e kushteve fillestare-kufitare të problemit. Duket gjithashtu se metoda e propozuar minimizon vështirësitë e procedimit algjebrik të metodës klasike të karakteristikave si dhe kompleksitetin shtjellues të saj.

INTRODUCTION

Overhead transmission lines, suspension bridges and many other objects known as flexibles structures can be subject of oscillations due to different causes. The mathematical models that describes these oscillations can be expressed in initial-boundary value problems for wave equations like in [3, 5, 6] or for string equations like in [1, 2]. The corresponding partial differential equations can be linear or nonlinear of second or fourth order with classical or non-classical boundary conditions.

The following model is derived and analyzed in [4] for the vibrations of a string which is fixed at $x = 0$ and is attached to a dashpot system at $x = \pi$:

Find the function u(x, t) which satisfies the equation

$$u''_{tt} - u''_{xx} = \varepsilon\left(u'_t - \frac{1}{3}u'^3_t\right), \quad 0 < x < \pi, \qquad t > 0 \tag{1}$$

Subject to boundary conditions

$$u(0,t) = 0, \qquad t \geq 0 \tag{2}$$

$$u'_x(\pi,t) = -\varepsilon\alpha u'_t(\pi,t), \qquad t \geq 0 \tag{3}$$

and initial conditions

$$u(x,0) = \phi(x), \qquad 0 < x < \pi \tag{4}$$

$$u'_t(x,0) = \psi(x), \qquad 0 < x < \pi \tag{5}$$

The above functions $\phi(x)$ *and* $\psi(x)$ are the initial displacement and the initial velocity of the string, the damping parameter α is a positive constant, and ε is a small dimensionless parameter ($0 < \varepsilon \ll 1$).

Thus, in (1-5) we have an initial-boundary value problem for a weakly nonlinear partial differential equation with a non-classical right boundary condition. It can be considered as a model describing the galloping oscillations of the overhead transmission lines in a wind field. In this case one of the aims of the study is to find the values of the damping parameter α for which the solution $u(x,t)$ tends to zero or tends to a certain bounded function.

As it is shown in [4] that the problem (1-5) is well-posed. The Laplace transform method is initially used to construct analytical approximation of the solution $u(x,t)$ for the linear variant of equation (1), which is obtained after neglecting the nonlinear term $\frac{1}{3}u'^3_t$. A two-timescales perturbation method is used only for the nonlinear case of simple initial conditions, referred as the monochromatic conditions of the form

$$\phi(x) = a_n \sin(nx) \quad \text{and} \quad \psi(x) = b_n \sin(nx)$$

Analytical solutions to these differential equations pose several practical difficulties: 1) analytical solution may be possible for restricted cases 2) analytical solution methods in most cases are very complicated 3) analytical solution found may be rather inconvenient for practical use 4) in most cases the problem must reinvestigated and resolved for any nonessential changing in its initial-boundary conditions.

An indirect numerical method for the solution of the problem (1-5) by transforming the second order PDE (1) to a system of two first order PDE$_s$ is presented in [4]. A difference scheme of the first order, which is supposed to have minimum accuracy, is then applied for the PDE$_s$ system.

The numerical method proposed in this paper for the solution of the problem (1-5) is based on the method of characteristics which is applicable for the weakly nonlinear hyperbolic problem of the general form:

$$a_1u''_{tt} + a_2u''_{tx} + a_3u''_{xx} = a_4, \quad 0 < x < l, \qquad t > 0, \tag{6}$$

where a_i denotes a function in the variables x, t, u, u'_x and u'_t for i = 1, 2, 3, 4.

The initial and boundary conditions are given as

$$u(0,t) = u(l,t) \equiv 0, \qquad t > 0 \tag{7)-(8}$$

$$u(x,0) = \phi(x), \qquad 0 \leq x \leq l \tag{9}$$

$$u'_t(x,0) = \psi(x), \qquad 0 \leq x \leq l \tag{10}$$

It can be seen that differential equation (1) is a special case of the equation (6), but the initial-boundary conditions (2-5) seem to be considerably different from the analogous conditions (7-10).

NUMERICAL SOLUTION FOR THE WAVE PROBLEM (1-4)

Let denote P(x_1, t_1) and Q(x_2, t_2) with t_1=t_2=0, as two points along x-axis so that $0 < x_1 < x_2 < \pi$. One can easily verify that the straight lines $t - t_1 = \pm x - x_1$ and $t - t_2 = \pm x - x_2$ are the 4 characteristic curves of the equation (1) in points P and Q respectively. Let $R(x,t)$ be the intersection of the proper lines $t - t_1 = x - x_1$ and $t - t_2 = -x - x_2$. The standard procedure of the method of characteristics for the determination of the point $R(x,t)$ and finding the approximations for $u(x,t), u'_x(x,t)$ *and* $u'_t(x,t)$ at this point (hereafter known as (P,Q,R) process), is very simplified for the case of equation (1). Since $u(x,t), u'_x(x,t)$ *and* $u'_t(x,t)$ are known at points P and Q, the conditions under which u''_{xx}, u''_{xy} *and* u''_{yy} can be uniquely found, after some algebraic operations, are obtained by the equations:

$$u'_x(R) = \frac{1}{2}[u'_x(Q) + u'_x(P) + u'_t(Q) - u'_t(P)] + \frac{h}{8}[a(Q) - a(P)] \quad (11)$$

$$u'_t(R) = \frac{1}{2}[u'_x(Q) - u'_x(P) + u'_t(Q) + u'_t(P)] + \frac{h}{8}[a(Q) + 2a(R) + a(P)] \quad (12)$$

with $h = (x_2 - x_1)$

It can be seen that the first equation of the simultaneous system (11) – (12) is an explicit one, while the second equation is implicit because of the term $a(R) = \varepsilon\left[u'_t(R) - \frac{1}{3}u'^3_t(R)\right]$ involved in it. This fact will be used latter.

In the same way the approximation to $u(R)$ results as,

$$u(R) = \frac{1}{2}[u(P) + u(Q)] + \frac{h}{8}[[u'_x(P) - u'_x(Q) + u'_t(P) + 2u]'_t(R) + u'_t(Q)] \quad (13)$$

It can be seen that (11) – (12) for finding the approximations to $u'_x(x,t)$ and $u'_t(x,t)$ is independent and autonomous from the process (13) for finding the approximation to $u(x,t)$, because the function $u(x,t)$ is not involved explicitely in equations (11) – (12). This fact will also be used latter.

A mesh G is obtained by discretizing the interval $0 \leq x \leq \pi$ into m subintervals each of width $h = \pi/m$. It is assumed that $\phi'(x)$ exists so that $u'_x(x,0) = \phi'(x)$, whenever $0<x<\pi$. Consequently, $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ will be known functions whenever $0 < x < \pi$ and $t = 0$. Meanwhile, following the conditions (2 – 5), one can easily obtain:

$$u(0,0) = 0, \qquad u'_t(0,0) = 0 \quad (14)$$

$$u'_x(\pi,0) + \varepsilon\alpha u'_t(\pi,0) = 0 \quad (15)$$

Considering the conditions (14) and (15) it is obvious that there are only three – out of the six necessary definitions of $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ at points x = 0 and x= π , there are only three of them. Thus, the method of characteristics can not be "initialized".

This difficulty can be overcome by the proposed method as it follows:

The notation G_0 will be used hereafter to denote the points of G. The process (P,Q,R) is applied first for (m-1) interior points of G_0 and so (m-2) points are obtained where the function u and its partial derivatives are approximated. The process (P,Q,R) is repeated for the last (m-2) points and so (m-3) other new points are obtained. It will be shown now how the values of $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ will be approximated at points A, D, F and S, T, V (see the figure 1). Let suppose that the values of u and its partial derivatives, $u(S), u'_x(S)$ and $u'_t(S)$, are known at point S. If the processes (P,Q,R) (Q,S,T) and (R,T,V) would be applied, then the approximations to $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ at point V would be obtained. The notations $u(V,h)), u_1x^{\dagger\prime}(V,h)$ and $u_1t^{\dagger\prime}(V,h)$ will be used to denote these approximations, to express the fact that the step h is used. Other approximations for $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ at point V would be obtained if the process (P,S,V) would be applied. The notations $u(V,2h), u'_x(V,2h)$ and $u'_t(V,2h)$ will be used to denote these last approximations, just to express the fact that the step 2h is used here. The two kinds of the approximations above would be obtained by two different ways, so that three equations can be written by equaling them value-to-value. The two equations corresponding to $u'_x(x,t)$ and $u'_t(x,t)$ are dependent, since the approximations (11) and (12) were obtained simultaneously, and, only the equation based on $u'_x(x,t)$ would be maintained (which is preferred due to its explicit form). Assembling the two independent remained equations with equation (15) provides a system of three equations for the determination of the approximations to $u(S), u'_x(S)$ and $u'_t(S)$.

The algebra would be rather complicated to write this procedure analytically, but it can be easily implemented numerically considering the above facts. The following algorithm details the procedure for the determination of $u(x,t), u'_x(x,t)$ and $u'_t(x,t)$ at points S, V, A and F.

Step 1 guess the value p for $u'_x(S)$ and compute the approximation to $u'_t(S)$ by (15)

Step 2 write equations (11) and (12) for points Q, S, T and compute the approximations to $u'_x(T)$ and $u'_t(T)$. Denote these approximations as $u'_x(T,p)$ and $u'_t(T,p)$ to express the fact that they depend on p.

Step 3 write only the explicit equation (11) for points R, T, V and compute the approximation to $u'_x(V)$. Denote this approximation as $u'_x(V,p,\boldsymbol{h})$.

Step 4 write only the explicit equation (11) for points P, S, V and compute the approximation to $u'_x(V)$. Denote this approximation as $u'_x(V,p,2\boldsymbol{h})$.

Step 5 solve iteratively the equation $d(p) = u'_x(V,p,2\boldsymbol{h}) - u'_x(V,p,\boldsymbol{h}) \equiv \boldsymbol{0}$ and find its root p^*. Set $u'_x(S) = p^*$ and from (15) compute $u'_t(S)$.

Step 6 As soon as $u'_x(S)$ *and* $u'_t(S)$ are determined, guess the value q *for* $u(S)$ and similarly to steps 1-5 find the value of q as the root of the equation $d(q) = u(V,q,2\boldsymbol{h}) - u(V,q,\boldsymbol{h}) \equiv \boldsymbol{0}$ At the end of this step the approximations to $u(x,t), u'_x(x,t)$ *and* $u'_t(x,t)$ at points S and V are found.

Step 7 Find the approximation to $u'_x(x,t)$ at points A and F by a similar but simpler process as above.
As it can be seen from the steps 1 to 5 only one implicit equation (12) must be solved per each iteration step 5.
Let us denote by G_1 the mesh of (m+1) points formed by points A_1, F, V and S_1 as in figure 1, and by (m-3) points obtained by the repeated (P,Q,R) process for interior points of G_0. It can be seen that based on the mesh G_1, a mesh G_2 may be constructed in the same way, and then a mesh G_3 and so on, in order to move farther up the time axis. So a uniform and square mesh of points is obtained where the function $u(x,t)$ and its partial derivatives are approximated. The full detailed algorithm of the proposed method, its implementation in MATLAB, and the comparison of numerical results with those appearing in literature will be presented in another publication.

CONCLUSIONS

Numerical solution has been developed in this paper for a wave equation with boundary damping. This problem can be regarded as a simple model describing oscillations of flexible structures such as overhead transmission lines in a wind field.
A numerical method of the type of characteristics was found and applied. It has been taken advantage of the particularities of the problems to be solved, namely the special configuration of the mesh of points generated by characteristics curves of the equation and the specialty of the initial-boundary conditions involved. As the original initial conditions of the problem did not provide sufficient data for the unknown function and its derivatives at the two endpoints, the classic method of characteristics could not be initialized. So this method was applied only for the interior points of a uniform mesh, but in a repeated way. An iterative technique was used then to compute the lacking data at the two endpoints, applying the method of characteristics for the 6 endpoints in two ways: as a double process with step h and then as a single process with double step 2h. The numerical method proposed here is easier in stating and coding compared to the complicated algebraic procedures introduced by the classic method of characteristics.

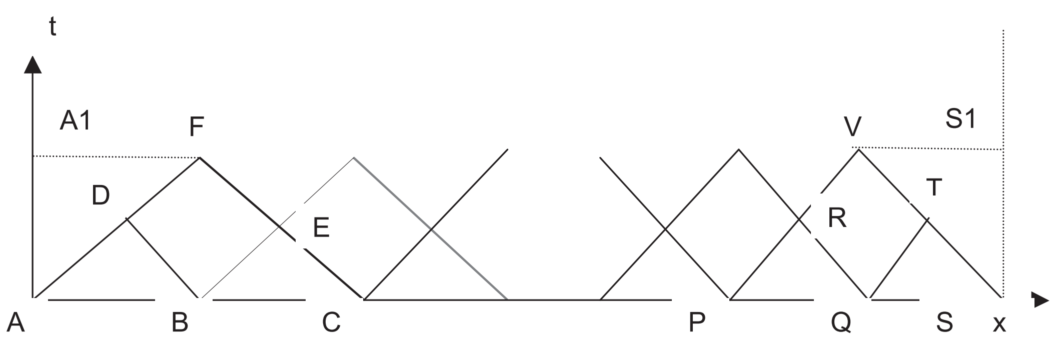

Figure 1: The characteristic curves of the equation (1) and the meshes G_0 and G_1

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