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ON REGULAR TERNARY SEMIHYPERGROUPS

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SUMMARY
This paper deals with a class of algebraic hypersystems which represent a generalization of semigroups and hypersemigroups. This class of hypersystems is called ternary semihypergroups. The notion of regularity of different type of algebraic systems has been introduced, studied and characterized by different authors such as Neumann, Iseki, Kovacs, Lajos etc. Different authors have studied the notion of regularity in ternary algebraic systems. In this paper we generalize this notion in ternary semihypergroups and we study some interesting properties of regular ternary semihypergroups, completely regular ternary semihypergroups, intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups.
Key words: ternary semihypergroup, hyperideal, completely-regular, intra-regular, regular, completely-semiprime.

1. INTRODUCTION AND PRELIMINARIES

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] who introduced the notion of "cubic matrix" which in turn was generalized by Kapranov, et al. in 1990 [5]. Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics and other sciences. The notion of an n-ary group was introduced in 1928 by W. Dörnte [2] (under inspiration of Emmy Noether). The idea of investigations of n-ary algebras, i.e., sets with one n-ary operation, seems to be going back to Kasner's lecture [4] in 1904. Different applications of ternary structures in physics are described by R. Kerner in [6]. The theory of ternary algebraic system was introduced by D. H. Lehmer [8] in 1932. The notion of ternary semigroups was introduced by S. Banach (cf. [10]).

Hyperstructure theory was introduced in 1934, when F. Marty [11] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

The notion of regularity was introduced and studied by J. von Neumann [12] in 1936. The notion of regularity of different type of algebraic systems has been characterized by different authors such as Iseki, Kovacs, Lajos [3,7,9] etc. Different authors have studied the notion of regularity in ternary algebraic systems. In this paper we generalize the notion of regularity in ternary semihypergroups and we study some interesting properties of regular, completely regular and intra-regular ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups.

Recall first the basic terms and definitions from the ternary semihypergroups theory.

Definition 1.1 A map $f : H \times H \times H \to \mathcal{P}^*(H)$ is called ternary hyperoperation on the set H, where H is a nonempty set and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subsets of H.

Definition 1.2 A ternary hypergroupoid is called the pair (H,f) where f is a ternary hyperoperation on the set H.

If A,B,C are nonempty subsets of H, then we define

$$f(A,B,C) = \bigcup_{a\in A, b\in B, c\in C} f(a,b,c).$$

Definition 1.3 A ternary hypergroupoid (H,f) is called a ternary semihypergroup if $\forall a_1, a_2, \ldots, a_5 \in H$, we have $f(f(a_1,a_2,a_3),a_4,a_5) = f(a_1,f(a_2,a_3,a_4),a_5) = f(a_1,a_2,f(a_3,a_4,a_5))$.

Definition 1.4 Let (H,f) be a ternary semihypergroup. Then H is called a ternary hypergroup if $\forall a,b,c \in H$, $\exists x,y,z \in H$ such that $c \in f(x,a,b) \cap f(a,y,b) \cap f(a,b,z)$.

Definition 1.5 Let (H,f) be a ternary hypergroupoid. Then (H,f) is commutative if $\forall a_1,a_2,a_3 \in H$ and $\forall \sigma \in S_3, f(a_1,a_2,a_3) = f(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3})$)

Definition 1.6 Let (H,f) be a ternary semihypergroup and T a nonempty subset of H. Then T is called a ternary subsemihypergroup of H if and only if $f(T,T,T) \subseteq T$.

Definition 1.7 A nonempty subset I of a ternary semihypergroup H is called a left (right, lateral) hyperideal of H if $f(H,H,I) \subseteq I (f(I,H,H) \subseteq I, f(H,I,H) \subseteq I)$.

A nonemtpy subset I of a ternary semihypergroup H is called a hyperideal of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of a ternary semihypergroup H is called two-sided hyperideal of H if it is a left and right hyperideal of H. A lateral hyperideal I of a ternary semihyperideal H is called a proper lateral hyperideal of H if $I \neq H$.

Definition 1.8 A left hyperideal I of a ternary semihypergroup H is called idempotent if $f(I,I,I) = I$.

Example 1.9 Let $H = \{a,b,c,d,e,g\}$ and $f(x,y,z) = (x*y)*z$, $\forall x,y,z \in H$, where $*$ is defined by the table:

*	a	b	c	d	e	g
a	a	{a,b}	c	{c,d}	e	{e,g}
b	b	b	d	d	g	g
c	c	{c,d}	c	{c,d}	c	{c,d}
d	d	d	d	d	d	d
e	e	{e,g}	c	{c,d}	e	{e,g}
g	g	g	d	d	g	g

Then (H,f) is a ternary semihypergroup. Clearly, $I_1 = \{c,d\}$, $I_2 = \{c,d,e,g\}$ and H are lateral hyperideals of H.

Example 1.10 Let $H = \{a,b,c,d,e,g\}$ and $f(x,y,z) = (x*y)*z$, $\forall x,y,z \in H$, where $*$ is defined by the table:

*	a	b	c	d	e	g
a	{b,c}	{b,c}	{b,c}	{b,c}	{b,c}	{b,c}
b	{a,c}	{a,c}	{a,c}	{a,c}	{a,c}	{a,c}
c	{a,b}	{a,b}	{a,b}	{a,b}	{a,b}	{a,b}
d	$H-d$	$H-d$	$H-d$	$H-d$	$H-d$	$H-d$
e	$H-e$	$H-e$	$H-e$	$H-e$	$H-e$	$H-e$
g	$H-g$	$H-g$	$H-g$	$H-g$	$H-g$	$H-g$

Then (H,f) is a ternary semihypergroup. There is no proper lateral hyperideal of H.

Example 1.11 Let $|H|\geq 4$ and $f:H\times H\times H\rightarrow \mathcal{P}^*(H)$, defined as follows:

$$f(x_0,x_0,x_0) = H-\{x_0,x_1\}$$
$$f(x,y,z) = H-\{x_0,x_2\}, \forall (x,y,z)\neq (x_0,x_0,x_0)$$

and $x_0\neq x_1\neq x_2\neq x_0$. (H,f) is a ternary semihypergroup. It can be seen that $H-\{x_0\}$ and $H-\{x_0,x_2\}$ are proper lateral hyperideals of H.

Example 1.12 Let $H=\{a,b,c\}$ be a set with a ternary hyperoperation f defined as follows:

$$f(x,y,z)=\begin{cases} x \text{ for } x=y=z, \\ b \text{ for } x\neq y\neq z, \\ z \text{ for } x=y, x\neq z, x\neq b, \\ \{a,c\} \text{ for } x=y=b, z\neq b. \end{cases}$$

It is easy to see that (H,f) is a ternary semihypergroup, and further it is a commutative ternary hypergroup.

It is clear that due to associative law in ternary semihypergroup H, $\forall x_1,x_2,\ldots,x_{2n+1}\in H$ and $m,n\in Z^+$ with $m\leq n$, one may write

$$f(x_1,x_2,\ldots x_{2n+1}) = f(x_1,\ldots x_m,x_{m+1},x_{m+2},\ldots x_{2n+1}) =$$
$$= f(x_1,\ldots,f(f(x_m,x_{m+1},x_{m+2}),x_{m+3},x_{m+4}),\ldots x_{2n+1}).$$

Let (H,f) be a ternary semihypergroup. It is clear that the intersection of all lateral hyperideals of a ternary subsemihypergroup T of H containing a nonempty subset A of T is the lateral hyperideal of H generated by A.

For every element $a\in H$, the left, right, lateral, two-sided and hyperideal generated by a are respectively given by

$$\langle a\rangle_l = \{a\}\cup f(H,H,a)$$
$$\langle a\rangle_r = \{a\}\cup f(a,H,H)$$
$$\langle a\rangle_m = \{a\}\cup f(H,a,H)\cup f(H,H,a,H,H)$$
$$\langle a\rangle_t = \{a\}\cup f(H,H,a)\cup f(a,H,H)\cup f(H,H,a,H,H)$$
$$\langle a\rangle = \{a\}\cup f(H,H,a)\cup f(a,H,H)\cup f(H,a,H)\cup f(H,H,a,H,H)$$

Definition 1.13 Let (H,f) be a ternary semihypergroup. A proper hyperideal P of H is called prime hyperideal of H if $f(A,B,C)\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$ or $C\subseteq P$ for any three hyperideals A,B,C of H.

Definition 1.14 Let (H,f) be a ternary semihypergroup. A proper hyperideal P of H is said to be strongly irreducible, if for hyperideals T and K of H, $T\cap K\subseteq P$ implies that $T\subseteq P$ or $K\subseteq P$

Definition 1.15 A proper hyperideal A of a ternary semihypergroup H is called a semiprime hyperideal of H if $f(I,I,I)\subseteq A$ implies $I\subseteq A$ for any hyperideal I of H.

Definition 1.16 A proper hyperideal A of a ternary semihypergroup H is called completely semiprime hyperideal of H if $f(x,x,x)\subseteq A$ implies that $x\in A, \forall x\in A$.

Definition 1.17 A subsemihypergroup B of a ternary semihypergroup H is called a bi-hyperideal of H if $f(B,H,B,H,B)\subseteq B$ **2.**

2. REGULAR TERNARY SEMIHYPERGROUPS

Definition 2.1 A ternary semihypergroup H is said to be regular if $\forall a\in H$, $\exists x\in H$ such that $a\in f(a,x,a)$.

A ternary semihypergroup H is called regular if all of its elements are regular.

It is clear that every ternary hypergroup is a regular ternary semihypergroup.

The ternary semihypergroups of the Examples 1.9 and 1.12 are regular ternary semihypergroups.

Lemma 2.2 Every lateral hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.

Proof. Let L be a lateral hyperideal of a regular ternary semihypergroup H. Then $\forall a \in L$, $\exists x \in H$, such that $a \in f(a,x,a)$. Now $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a)) \subseteq f(a,f(x,a,x),a) \subseteq f(a,L,a)$. So $\exists b \in L$ such that $a \in f(a,b,a)$. This implies that L is a regular ternary semihypergroup.

Remark. Every hyperideal of a regular ternary semihypergroup H is a regular ternary semihypergroup.

Theorem 2.3 Let (H,f) be a ternary semihypergroup. Then the following statements are equivalent:

1. H is regular.
2. For any right hyperideal R, lateral hyperideal M and left hyperideal L of H, $f(R,M,L) = R \cap M \cap L$.
3. $\forall a,b,c \in H$, $f(\langle a\rangle_r, \langle b\rangle_m, \langle c\rangle_l) = \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$.
4. $\forall a \in H$, $f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l) = \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$.

Proof. $(1) \Rightarrow (2)$. Let H be a regular ternary semihypergroup. Let R,M and L be a right, a lateral and a left hyperideal of H respectively. Then clearly, $f(R,M,L) \subseteq R \cap M \cap L$. Now for $a \in R \cap M \cap L$, we have $a \in f(a,x,a)$ for some $x \in H$. This implies that $a \in f(a,x,a) \subseteq f(f(a,x,a),x,f(a,x,a)) \subseteq f(R,M,L)$. Thus we have $R \cap M \cap L \subseteq f(R,M,L)$. So we find that $f(R,M,L) = R \cap M \cap L$.

Clearly, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$. Let $a \in H$. Clearly, $a \in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = f(\langle a\rangle_r, \langle a\rangle_m, \langle a\rangle_l)$. Then we have, $a \in f(f(a,H,H) \cup \{a\}, f(H,a,H) \cup f(H,H,a,H,H) \cup \{a\}, f(H,H,a) \cup \{a\}) \subseteq f(a,H,a)$. So we find that $a \in f(a,H,a)$ and hence $\exists x \in H$ such that $a \in f(a,x,a)$. This implies that a is regular and hence H is regular.

Corollary 2.4 Let (H,f) be a ternary semihypergroup. Then the following statements are equivalent:

1. H is regular.
2. For any right hyperideal R and left hyperideal L of H, $f(R,H,L) = R \cap L$.
3. $\forall a,b \in H$, $f(\langle a\rangle_r, H, \langle b\rangle_l) = \langle a\rangle_r \cap \langle b\rangle_l$.
4. $\forall a \in H$, $f(\langle a\rangle_r, H, \langle a\rangle_l) = \langle a\rangle_r \cap \langle a\rangle_l$.

Theorem 2.5 A ternary semihypergroup H is regular if and only if every hyperideal of H is idempotent.

Proof. Let H be a regular ternary semihypergroup and I be any hyperideal of H. Then $f(I,I,I) \subseteq f(H,H,I) \subseteq I$. Let $a \in I$. Then $\exists x \in H$ such that $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a))$. Since I is a hyperideal and $a \in I$, $f(x,a,x) \subseteq I$. Thus $a \in f(a,x,a) \subseteq f(a,x,f(a,x,a)) \subseteq f(I,I,I)$. Consequently, $I \subseteq f(I,I,I)$ and hence $f(I,I,I) = I$, that is I is idempotent.

Conversely, suppose that every hyperideal of H is idempotent. Let A,B and C be three hyperideals of H. Then $f(A,B,C) \subseteq f(A,H,H) \subseteq A$, $f(A,B,C) \subseteq f(H,B,H) \subseteq B$ and $f(A,B,C) \subseteq f(H,H,C) \subseteq C$. This implies that $f(A,B,C) \subseteq A \cap B \cap C$. Also, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) \subseteq f(A,B,C)$. Again, since $A \cap B \cap C$ is a hyperideal of H, $f(A \cap B \cap C, A \cap B \cap C, A \cap B \cap C) = A \cap B \cap C$. Thus $A \cap B \cap C \subseteq f(A,B,C)$ and hence $A \cap B \cap C = f(A,B,C)$. Therefore, by Theorem 2.3, H is a regular ternary semihypergroup.

Theorem 2.6 A commutative ternary semihypergroup H is regular if and only if every hyperideal of H is semiprime.

Proof. Let H be a commutative regular ternary semihypergroup and I be any hyperideal of H such that $f(A,A,A)\subseteq I$ for any hyperideal A of H. From Theorem 2.3, it follows that $f(A,A,A)=A$. Consequently, $A\subseteq I$ and hence I is a semiprime hyperideal of H.

Conversely, suppose that every hyperideal of a commutative ternary semihypergroup H is semiprime. Let $a\in H$. Then $f(a,H,a)$ is a hyperideal of H. Now by hypothesis, $f(a,H,a)$ is a semiprime hyperideal of H. If $f(a,H,a)=H$, then we are done. Now suppose that $f(a,H,a)\neq H$. Then

$$\begin{aligned} f(\langle a\rangle,\langle a\rangle,\langle a\rangle) &= f(f(H,H,a)\cup f(a,H,H)\cup f(H,a,H)\cup \\ &\quad \cup f(H,H,a,H,H)\cup\{a\}, f(H,H,a)\cup f(a,H,H)\cup \\ &\quad \cup f(H,a,H)\cup f(H,H,a,H,H)\cup\{a\}, f(H,H,a)\cup \\ &\quad \cup f(a,H,H)\cup f(H,a,H)\cup f(H,H,a,H,H)\cup\{a\}) \\ &\subseteq f(a,H,a) \end{aligned}$$

that is, $f(\langle a\rangle,\langle a\rangle,\langle a\rangle)\subseteq f(a,H,a)$. This implies that $\langle a\rangle\subseteq f(a,H,a)$, since $f(a,H,a)$ is a semiprime hyperideal of H. Consequently, $a\in f(a,x,a)$ for some $x\in H$ and hence H is a regular ternary semihypergroup.

Theorem 2.7 Let (H,f) be a ternary semihypergroup and I a hyperideal of H. The following statements are equivalent:

1. I is a regular hyperideal of H;
2. $\forall a\in H$, $I\cup f(\langle a\rangle_r,\langle a\rangle_m,\langle a\rangle_l)=I\cup(\langle a\rangle_r\cap\langle a\rangle_m\cap\langle a\rangle_l)$;
3. $\forall a\in H\setminus I$, either $a\in f(a,a_1,a,a_2,a)$ or $a\in f(a,b_1,b_2,a,b_3,b_4,a)$, for some $a_1,a_2,b_1,b_2,b_3,b_4\in H$.

Proof. $(1)\Rightarrow(2)$. Suppose that I is a regular hyperideal. Then $\forall a\in H$,

$I\cup(\langle a\rangle_r\cap\langle a\rangle_m\cap\langle a\rangle_l)\subseteq\langle I\cup\langle a\rangle_r\rangle_r,\langle I\cup\langle a\rangle_m\rangle_m,\langle I\cup\langle a\rangle_l\rangle_l$ Moreover, since each of the three sets on the right side contains I, then we have

$$\begin{aligned} &I\cup(\langle a\rangle_r\cap\langle a\rangle_m\cap\langle a\rangle_l)\subseteq \\ &\subseteq\langle I\cup\langle a\rangle_r\rangle_r\cap\langle I\cup\langle a\rangle_m\rangle_m\cap\langle I\cup\langle a\rangle_l\rangle_l= \\ &=I\cup f(I\cup\langle a\rangle_r,I\cup\langle a\rangle_m,I\cup\langle a\rangle_l)=I\cup f(I,I\cup\langle a\rangle_m,I\cup\langle a\rangle_l)\cup \\ &\quad\cup f(\langle a\rangle_r,I,I\cup\langle a\rangle_l)\cup f(\langle a\rangle_r,\langle a\rangle_m,I)\cup f(\langle a\rangle_r,\langle a\rangle_m,\langle a\rangle_l)= \\ &=I\cup f(\langle a\rangle_r,\langle a\rangle_m,\langle a\rangle_l)\subseteq I\cup(\langle a\rangle_r\cap\langle a\rangle_m\cap\langle a\rangle_l). \end{aligned}$$

$(2)\Rightarrow(3)$. We note that

$$\begin{aligned} \langle I\cup\langle a\rangle_r\rangle_r &= \langle I\cup\langle a\rangle_r\rangle_r\cap H\cap H=I\cup f(\langle I\cup\langle a\rangle_r\rangle_r,H,H)= \\ &= I\cup f(I,H,H)\cup f(\langle a\rangle_r,H,H)\cup f(I,H,H,H,H)\cup \\ &\quad\cup f(\langle a\rangle_r,H,H,H,H)= \\ &= I\cup f(I,H,H)\cup f(a,H,H,H)\cup f(a,H,H,H,H)\cup \\ &\quad\cup f(I,H,H,H,H)\cup f(a,H,H,H,H)\cup f(a,H,H,H,H,H,H)= \\ &= I\cup f(I,H,H)\cup f(a,H,H)\cup f(a,H,H,H,H)= \\ &= \langle I\cup f(a,H,H)\rangle_r=I\cup f(a,H,H). \end{aligned}$$

In the same manner, we obtain

$\langle I\cup\langle a\rangle_m\rangle_m = \langle I\cup f(H,a,H)\rangle_m = I\cup f(H,a,H)\cup f(H,H,a,H,H),$

$\langle I\cup\langle a\rangle_l\rangle_l = \langle I\cup f(H,H,a)\rangle_l = I\cup f(H,H,a).$

Then

$$\langle I\cup f(a,H,H)\rangle_r \cap \langle I\cup f(H,a,H)\rangle_m \cap \langle I\cup f(H,H,a)\rangle_l =$$
$$= I\cup f(\langle I\cup f(a,H,H)\rangle_m, \langle I\cup f(H,a,H)\rangle_m, \langle I\cup f(H,H,a)\rangle_l) =$$
$$= I\cup f(a,H,H,H,a,H,H,H,a)\cup f(a,H,H,H,H,a,H,H,H,H,a) =$$
$$= I\cup f(a,H,a,H,a)\cup f(a,H,H,a,H,H,a).$$

The result now follows.

$(3)\Rightarrow(1)$. Let R be an arbitrary right hyperideal, M an arbitrary lateral hyperideal, L an arbitrary left hyperideal of H all containing I. Let us assume that I satisfies (3). It is clear that, $I\cup f(R,M,L)\subseteq R\cap M\cap L$.

Let $a\in R\cap M\cap L$. By (3), $a\in I$ or $a\in f(a,a_1,a,a_2,a)$ or $a\in f(a,b_1,b_2,a,b_3,b_4,a)$ for some $a_1,a_2,b_1,b_2,b_3,b_4\in H$. We note also that in the second and third cases we have:

$$a \in f(a,a_1,a,a_1,a,a_2,a,a_2,a) = f(f(a,a_1,a_2),f(a_1,a,a_2),f(a,a_2,a)),$$
$$a \in f(a,b_1,b_2,a,b_1,b_2,a,b_3,b_4,a,b_3,b_4,a) =$$
$$= f(f(a,b_1,b_2),f(a,b_1,b_2),a,f(b_3,b_4,a),f(b_3,b_4,a)).$$

Hence in the last two cases we have

$$a\in f(f(a,x_2,x_3),f(y_1,a,y_3),f(z_1,z_2,a)),$$

for some $x_2,x_3,y_1,y_2,z_1,z_2\in H$. Whence, in any case we have $a\in I\cup f(R,M,L)$ and therefore $I\cup f(R,M,L)=R\cap M\cap L$.

Theorem 2.8 Let (H,f) be a ternary semihypergroup and I a regular hyperideal of H. Then, for any right hyperideal R, lateral hyperideal M and left hyperideal L of H, if $f(R,M,L)\subseteq I$, then $R\cap M\cap L\subseteq I$.

Proof. Suppose $f(R,M,L)\subseteq I$ and I is a regular hyperideal. Then

$$R\cap M\cap L\subseteq \langle I\cup R\rangle_r \cap \langle I\cup M\rangle_m \cap \langle I\cup L\rangle_l =$$
$$I\cup f(\langle I\cup R\rangle_r, \langle I\cup M\rangle_m, \langle I\cup L\rangle_l) = I\cup f(I, \langle I\cup M\rangle_m, \langle I\cup L\rangle_l) =$$
$$= f(R,I,\langle I\cup L\rangle_l)\cup f(R,M,I)\cup f(R,M,L)\subseteq I.$$

Corollary 2.9 A regular and strongly irreducible hyperideal is always prime.

Corollary 2.10 Every regular hyperideal is prime.

Definition 2.11 Let (H,f) be a ternary semihypergroup and $Q\subseteq H$. Then Q is called a quasi-hyperideal of H if and only if

$$f(Q,H,H)\cap f(H,Q,H)\cap f(H,H,Q)\subseteq Q \text{ and } f(Q,H,H)\cap f(H,H,Q,H,H)\cap f(H,H,Q)\subseteq Q.$$

Theorem 2.12 Let (H,f) be a regular ternary semihypergroup and $Q\subseteq H$. Then Q is a quasi-hyperideal if and only if $f(Q,H,Q,H,Q)\cap f(Q,H,H,Q,H,H,Q)\subseteq Q$.

Proof. Let H be a regular ternary semihypergroup and Q be a quasi-hyperideal of H. Then $f(Q,H,Q,H,Q)\cap f(Q,H,H,Q,H,H,Q)\subseteq f(H,H,Q)$, $f(Q,H,H)$, and $f(H,Q,H)\cup f(H,H,Q,H,H)$ and hence

$$f(Q,H,Q,H,Q)\cap f(Q,H,H,Q,H,H,Q)\subseteq$$
$$\subseteq f(H,H,Q)\cap (f(H,Q,H)\cup f(H,H,Q,H,H))\cap f(Q,H,H)\subseteq Q.$$

Conversely, suppose that H is regular and $f(Q,H,Q,H,Q)\cap f(Q,H,H,Q,H,H,Q)\subseteq Q$.

Then

$$f(Q,H,H)\cap(f(H,Q,H)\cup f(H,H,Q,H,H))\cap f(H,H,Q)=$$
$$=f(f(Q,H,H),f(H,Q,H)\cup f(H,H,Q,H,H),f(H,H,Q))=$$
$$=f(f(Q,H,H),f(H,Q,H),f(H,H,Q))\cup f(f(Q,H,H),f(H,H,Q,H,H),$$
$$f(H,H,Q))\subseteq f(Q,H,Q,H,Q)\cup f(Q,H,H,Q,H,H,Q)\subseteq Q.$$

Theorem 2.13 Let (H,f) be a regular ternary semihypergroup and Q_1,Q_2,Q_3 be three quasi-hyperideals of H. Then $f(Q_1,Q_2,Q_3)$ is a quasi-hyperideal.

Proof.

$$f(f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3),H,f(Q_1,Q_2,Q_3))\cup f(f(Q_1,Q_2,Q_3),H,H,$$
$$f(Q_1,Q_2,Q_3),H,H,f(Q_1,Q_2,Q_3))=(f(Q_1,f(Q_2,Q_3,H),Q_1,f(Q_2,Q_3,H),Q_1),Q_2,Q_3)\cup$$
$$\cup f(f(Q_1,f(Q_2,Q_3,H),H,Q_1,f(Q_2,Q_3,H),H,Q_1),Q_2,Q_3)\subseteq(Q_1,Q_2,Q_3).$$

Corollary 2.14 The family of all quasi-hyperideals of a regular ternary semihypergroup is a ternary semihypergroup.

3. COMPLETELY REGULAR AND INTRA-REGULAR TERNARY SEMIHYPERGROUPS

Definition 3.1 Let (H,f) be a ternary semihypergroup. An element $a\in H$ is said to be left (resp. right) regular if $\exists x\in H$ such that $a\in f(x,a,a)$ (resp. $a\in f(a,a,x)$).

If all the elements of a ternary semihypergroup H are left (resp. right) regular, then H is called left (resp. right) regular.

The ternary semihypergroup of the Examples 1.9 is a completely regular ternary semihypergroup.

Theorem 3.2 A ternary semihypergroup (H,f) is left (resp. right) regular if and only if every left (resp. right) hyperideal of H is completely semiprime.

Proof. Let H be a left regular ternary semihypergroup and L be any left hyperideal of H. Suppose that $f(a,a,a)\subseteq L$ for $a\in H$. Since H is left regular, $\exists x\in H$ such that $a\in f(x,a,a)\subseteq f(x,f(x,a,a),a)\subseteq f(x,x,f(a,a,a))\subseteq f(H,H,L)\subseteq L$. Thus L is completely semiprime.

Conversely, suppose that every left hyperideal of H is completely semiprime. Now $\forall a\in H$, $f(H,a,a)$ is a left hyperideal of H. Then by hypothesis, $f(H,a,a)$ is a completely semiprime hyperideal of H. Now $f(a,a,a)\subseteq f(H,a,a)$. Since $f(H,a,a)$ is completely semiprime, it follows that $a\in f(H,a,a)$. So $\exists x\in H$ such that $a\in f(x,a,a)$. Consequently, a is left regular. Since a is arbitrary, it follows that H is left regular.

Similarly, it can be proved the theorem for the right regularity.

Definition 3.3 Let (H,f) be a ternary semihypergroup. An element $a\in H$ is said to be completely regular if it is left regular, right regular and regular.

If all the elements of H are completely regular, then H is called completely regular.

Proposition 3.4 A ternary semihypergroup (H,f) is completely regular if and only if $a\in f(a,a,H,a,a)$, $\forall a\in H$.

Proof. Let H be a completely regular ternary semihypergroup and $a\in H$. Then, by the definition, we have that $a\in f(a,a,H)$ and $a\in f(H,a,a)$, that is $a\in f(a,a,H)\cap f(H,a,a)$. Since H is completely regular, $\exists x\in H$ such that $a\in f(a,x,a)$. So we have

$$a\in f(a,x,a)\subseteq f(f(a,a,H),x,f(H,a,a))\subseteq f(a,a,f(H,x,H),a,a)\subseteq f(a,a,H,a,a).$$

Conversely, suppose that $\forall a\in H$, $a\in f(a,a,H,a,a)$. Then

1. $a\in f(a,a,H,a,a)\subseteq f(a,f(a,H,a),a)\subseteq f(a,H,a)$, that is H is regular.

2. $a \in f(a,a,H,a,a) \subseteq f(f(a,a,H),a,a) \subseteq f(H,a,a)$, that is H is left regular.

3. $a \in f(a,a,H,a,a) \subseteq f(a,a,f(H,a,a)) \subseteq f(a,a,H)$, that is H is right regular. Therefore H is completely regular.

Theorem 3.5 A ternary semihypergroup (H,f) is completely regular if and only if every bi-hyperideal of H is completely semiprime.

Proof. Suppose that H is completely regular ternary semihypergroup. Let B be any bi-hyperideal of H. Let $f(b,b,b) \subseteq B$ for $b \in B$. Since H is completely regular, from Proposition 3.4, it follows that $b \in f(b,b,H,b,b)$. This implies that $\exists x \in H$ such that

$$\begin{aligned} b &\in f(b,b,x,b,b) \subseteq f(b,f(b,b,x,b,b),x,f(b,b,x,b,b),b) = \\ &= f(b,b,b,f(x,b,b,x),b,f(b,b,x,b,b),x,b,b,b) = \\ &= f(b,b,b,f(x,b,b,x),b,b,b,f(x,b,b,x),b,b,b) \subseteq f(B,H,B,H,B) \subseteq B. \end{aligned}$$

This shows that B is completely semiprime.

Conversely, suppose that every bi-hyperideal of H is completely semiprime. Since every left and right hyperideal of a ternary semihypergroup H is a bi-hyperideal of H, it follows that every left and right hyperideal of H is completely semiprime. Consequently, we have from Theorem 3.2 that H is both left and right regular.

Let $a \in H$. We consider $f(a,H,a)$. Let $x,y,z \in f(a,H,a)$ and $h_1,h_2 \in H$. Then for some $h_0,h_0',h_0'' \in H$ we have:

$$f(x,h_1,y,h_2,z) \subseteq f(f(a,h_0,a),h_1,f(a,h_0',a),h_2,f(a,h_0'',a)) \subseteq f(a,f(h_0,a,h_1,a,h_0',a,h_2,a,h_0''),a) \subseteq f(a,H,a).$$

This implies that $f(f(a,H,a),H,f(a,H,a),H,f(a,H,a)) \subseteq f(a,H,a)$. That is, $f(a,H,a)$ is a bi-hyperideal of H. Since $f(a,a,a) \subseteq f(a,H,a)$ and $f(a,H,a)$ is completely semiprime, it follows that $a \in f(a,H,a)$, $\forall a \in H$. That is H is regular. This completes the proof.

Theorem 3.6 If (H,f) is a completely regular ternary semihypergroup, then every bi-hyperideal of H is idempotent.

Proof. Let H be a completely regular ternary semihypergroup and B be a bi-hyperideal of H. Since H is a completely regular ternary semihypergroup, it is also a regular ternary semihypergroup. Let $b \in B$. Then $\exists x \in H$ such that $b \in f(b,x,b)$. This implies that $b \in f(B,H,B)$ and hence $B \subseteq f(B,H,B)$. Also $f(B,H,B) \subseteq f(B,H,B,H,B) \subseteq B$. Thus we find that $B = f(B,H,B)$. Again, we have from Proposition 3.4 that $b \in f(b,b,H,b,b) \subseteq f(B,B,H,B,B)$. This implies that $B \subseteq f(B,B,H,B,B) = f(B,f(B,H,B),B) = f(B,B,B) \subseteq B$. Consequently, $f(B,B,B) = B$.

Definition 3.7 A ternary semihypergroup (H,f) is called intra-regular if $\forall a \in H, \exists x,y \in H$ such that $a \in f(x,a,a,a,y)$.

Theorem 3.8 If (H,f) is an intra-regular ternary semihypergroup, then for every left hyperideal L, lateral hyperideal M and right hyperideal R of H, $L \cap M \cap R \subseteq f(L,M,R)$.

Proof. Suppose that H is an intra-regular ternary semihypergroup. Let L,M and R be a left hyperideal, lateral hyperideal and a right hyperideal of H respectively. Now for $a \in L \cap M \cap R$, we have $a \in f(x,a,a,a,y)$ for some $x,y \in H$. This implies that $a \in f(x,a,a,a,y) \subseteq f(f(x,x,a,a,a),f(y,x,a,a,a,y,x),f(a,a,a,y,y)) \subseteq f(L,M,R)$. Thus we have $L \cap M \cap R \subseteq f(L,M,R)$.

Proposition 3.9 Let (H,f) be an intra-regular ternary semihypergroup. Then a non-empty subset I of H is a hyperideal of H if and only if I is a lateral hyperideal of H.

Proof. Clearly, if I is a hyperideal of H, then I is a lateral hyperideal of H.

Conversely, let I be a lateral hyperideal of an intra-regular ternary semihypergroup. Let $a \in I$ and $s,t \in H$. Then $a \in H$ and hence $\exists x,y \in H$ such that $a \in f(x,a,a,a,y)$. Now
$f(s,t,a) \subseteq f(s,t,f(x,a,a,a,y)) \subseteq f(H,I,H) \subseteq I$ and $f(a,s,t) \subseteq f(f(x,a,a,a,y),s,t) \subseteq f(H,I,H) \subseteq I$. This implies that I is both a left hyperideal and a right hyperideal of H. Consequently, I is a hyperideal of H.

Lemma 3.10 Every lateral hyperideal of an intra-regular ternary semihypergroup (H,f) is an intra-regular ternary semihypergroup.

Proof. Let L be a lateral hyperideal of an intra-regular ternary semihypergroup H. Then $\forall a \in L$, $\exists x,y \in H$ such that $a \in f(x,a,a,a,y)$. Now

$$a \in f(x,a,a,a,y) \subseteq f(x,f(x,a,a,a,y),f(x,a,a,a,y),f(x,a,a,a,y),y) \subseteq f(x,x,a,a,a,y,y),f(a,a,a),f(y,x,a,a,a,y,y)) \subseteq f(L,f(a,a,a),L)$$

This implies that $\exists u,v \in L$ such that $a \in f(u,f(a,a,a),v)$. Consequently, L is an intra-regular ternary semihypergroup.

From the Proposition 3.9 we have the following corollary:

Corollary 3.11 Every hyperideal of an intra-regular ternary semihypergroup H is an intra-regular ternary semihypergroup.

Theorem 3.12 Let I be a hyperideal of an intra-regular ternary semihypergroup H and J be a hyperideal of I. Then J is a hyperideal of the entire ternary semihypergroup H.

Proof. It is sufficient to show that J is a lateral hyperideal of H. Let $a \in J \subseteq I$ and $s,t \in H$. Then $f(s,a,t) \subseteq I$. We have to show that $f(s,a,t) \subseteq J$. From Corollary 3.11, it follows that I is an intra-regular ternary semihypergroup. Thus $\exists u,v \in I$ such that

$$f(s,a,t) \subseteq f(u,f(s,a,t),f(s,a,t),f(s,a,t),v) \subseteq f(f(u,s,a,t,s),a,f(t,s,a,t,v)) \subseteq f(I,J,I) \subseteq J$$

Consequently, J is a lateral hyperideal of H.

Theorem 3.13 A ternary semihypergroup (H,f) is intra-regular if and only if every hyperideal of H is completely semiprime.

Proof. Let H be an intra-regular ternary semihypergroup and I be a hyperideal of H. Let $f(a,a,a) \subseteq I$ for $a \in H$. Since H is intra-regular, $\exists x,y \in H$ such that $a \in f(x,f(a,a,a),y) \subseteq I$. Consequently, I is completely semiprime.

Conversely, suppose that every hyperideal of H is completely semiprime. Let $a \in H$. Then $f(a,a,a) \subseteq \langle f(a,a,a) \rangle$. This implies that $a \in \langle f(a,a,a) \rangle$, since $\langle f(a,a,a) \rangle$ is completely semiprime. Now

$$\langle f(a,a,a) \rangle = f(H,H,f(a,a,a)) \cup f(f(a,a,a),H,H) \cup f(H,f(a,a,a),H) \cup$$
$$\cup f(H,H,f(a,a,a),H,H) \cup f(a,a,a)$$

So we have the following cases: If $a \in f(H,H,f(a,a,a))$, then $f(a,a,a) \subseteq f(H,H,f(a,a,a),a,a)$. Hence
$a \in f(H,H,H,H,f(a,a,a),a,a) \subseteq f(H,H,H,a,a,a,H) \subseteq f(H,f(a,a,a),H)$ If $a \in f(f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,a,f(a,a,a),H,H)$.
Hence $a \in f(a,a,f(a,a,a),H,H,H,H) \subseteq f(H,a,a,a,H,H,H) \subseteq f(H,f(a,a,a),H)$ If $a \in f(H,f(a,a,a),H)$, then we are done.

If $a \in f(H,H,f(a,a,a),H,H)$, then $f(a,a,a) \subseteq f(a,H,H,f(a,a,a),H,H,a)$.

Hence, $a \in f(H,H,a,H,H,f(a,a,a),H,H,a,H,H) \subseteq f(H,H,H,f(a,a,a),H,H,H) \subseteq f(H,f(a,a,a),H)$.

If $a \in f(a,a,a)$, then, $a \in f(a,a,a) \subseteq f(f(a,a,a),f(a,a,a),f(a,a,a)) \subseteq f(H,f(a,a,a),H)$

So we find that in any case, H is intra-regular

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