ALGEBRA OF POLAR SUBSETS OF AN ORDERED SEMIGROUP

Kostaq HILA

Department of Mathematics & Computer Sciences,
Faculty of Natural Sciences, University of Gjirokastra,
ALBANIA
E-mail: *kostaq_hila@yahoo.com*

Abstract

The algebra of polar subsets of a *le*– and *poe*-semigroup is considered. It is considered the set $\mathcal{B}$ of the subsets A of S which are equal to the bipolar of A. It is proved that $\mathcal{B}$, partially ordered by set inclusion is a complete Boolean algebra.

Key words: *poe*-semigroup, *le*-semigroup, polar, bipolar, boolean algebra.

Përmbledhje

Në këtë punim është shqyrtuar algjebra e nënbashkësive polare të një *le*- dhe *poe*-gjysmëgrupi. Shqyrtohet bashkësia $\mathcal{B}$ e nënbashkësive A të S të cilat janë të barabarta me bipolarin e A. Provohet që $\mathcal{B}$, pjesërisht e renditur nga përfshirja e bashkësive, është një algjebër e plotë e Bulit.

1. Introduction and preliminaries

The ordered semigroups (:*po*-semigroups) and *l*-semigroups have been studied by a lot of mathematicians. The concept of *l*-semigroup is due to Fuchs [4] and G. Birkhoff [2]. Birkhoff arose a problem (prob.123, pp. 346) concerning the study of a special class of *l*-grupoids. This problem has been studied by different mathematicians for *m*-distributive lattices such as Choudhury [3] and Kehayopulu [4]. A part of those results have been extended and generalized recently in v*e*-Γ-semigroups [5]. In this paper we are dealing with *poe*- and *le*-semigroups. Motivation for this paper has been provided by several results obtained in [1, 6, 7, 8, 9]. We study the algebra of polar subsets of a *le*-semigroup and *poe*-semigroup. The results of this paper can serve as an object of a further study, in extending and generalizing them in ordered Γ-semigroups, which are widely studied recently by the author and other authors in several papers. Firstly, we introduce the concept of α_{x_0} -disjoint (resp. x_0-prime) elements of a *poe*-semigroup (resp. *le*-semigroup) and some related properties are investigated. In section 3 we introduce the concept of polar and bipolar of a subset A of a *le*-semigroup S and investigate different properties concluding with the main theorem in which we show that the set $\mathcal{B}$ of the subsets A of S which are equal to the bipolar of A, partially ordered by set inclusion, is a complete Boolean algebra. In section 4 we put an equivalence relation $\approx$ in a distributive *le*-semigroup and we prove that the set of corresponding equivalence classes S is a distributive *le*-semigroup and there exists an isomorphism between S and $\mathcal{B}$. In the last section, we study, investigate and generalize the same properties in *poe*-semigroups.

An ordered semigroup (: po-semigroup) is an ordered set $(S,\leq)$ at the same time a semigroup such that for all $a,b,x\in S$, $a\leq b \Rightarrow xa\leq xb$ and $ax\leq bx$.

A poe-semigroup is a po-semigroup having a greatest element e.

A *le-semigroup* $(S;\cdot,\vee,\wedge)$ is an algebra defined as follows:

1. $\langle S;\cdot\rangle$ is a semigroup;
2. $\langle S;\vee;\wedge\rangle$ is a lattice with a greatest element which is denoted by e throughout this paper;
3. for all $a,b,c\in S$, $a(b\vee c)=ab\vee ac$ and $(b\vee c)a=ba\vee ca$.

When S is a semilattice under $\vee$ with a greatest element e, it is called $\vee e$-semigroup.

2. x_0-PRIME ELEMENTS OF le-SEMIGROUPS

Let S be a poe-semigroup and $x_0\in S$. We shall say that the elements $a,b\in S$ are α*-disjoint with respect to* x_0 (or α_{x_0}*-disjoint*) denoted by $a\mathrm{T}_{x_0}b$, if and only if for every element $x\in S$ with $x\geq a$ and $x\geq b$, implies $x=x_0$ [7].

Let S be a le-semigroup and $x_0\in S$. We shall say that the elements $a,b\in S$ are *prime with respect to* x_0 (or $x_0-prime$) denoted by $a\Pi_{x_0}b$, if $a\vee b=x_0$ [7].

The elements $a_\alpha,\alpha\in A$ of a poe-semigroup (resp. le-semigroup) S will be called pairwise α_{x_0}-disjoint (resp. pairwise x_0-prime) if and only if $a_i\mathrm{T}_{x_0}a_j$ (resp. $a_i\Pi_{x_0}a_j$) for every $i,j\in A$ with $i\neq j$ [7].

The following proposition holds true.

Proposition 2.1. *Let S be a le-semigroup, $x_0\in S$ and $a,b\in S$. The following statements are equivalent:*

1. $a\mathrm{T}_{x_0}b$.
2. $a\Pi_{x_0}b$ and $x_0=e$.

Proof. $1.\Rightarrow 2.$ Let $a\mathrm{T}_{x_0}b$. It is clear that this implies $a\Pi_{x_0}b$. Let now $a\mathrm{T}_{x_0}b$ and $y\in S$, then, since $a\Pi_{x_0}b$, we have $x_0\geq a,b$. If $z=y\vee x_0$, then $z\geq a,b$. Therefore $z=x_0$. Thus we obtain $x_0\geq y$, that is, $x_0=e$.

$2.\Rightarrow 1.$ Let $x\in S, x\geq a,b$. Then, since 2. is true, we have $x\geq a\vee b=x_0$ and $x\leq x_0$. Therefore $x=x_0$. This completes the proof.

The following proposition is an immediately corollary of the above proposition.

Corollary 2.2. *Let S be a le-semigroup and $x_0\in S$. If there exist the elements $a,b\in S$ such that $a\mathrm{T}_{x_0}b$, then for every $x,y\in S$, $x\mathrm{T}_{x_0}y$ if and only if $x\Pi_{x_0}y$.*

Proposition 2.3. *Let S be a poe-semigroup and $x_0\in S$. The following properties hold true:*

1. if $a\mathrm{T}_{x_0}b$, then $b\mathrm{T}_{x_0}a$.
2. if $a\mathrm{T}_{x_0}b$ and $a\leq b$, then $b=x_0$.
3. if $a\mathrm{T}_{x_0}b, a\leq a'$ and $b\leq b'$, then $a'\mathrm{T}_{x_0}b'$.
4. if the elements $a_i\in S, i\in I$ are pairwise α_{x_0}-disjoint and if for the elements $b_i\in S, i\in I, a_i\leq b_i$, for all $i\in I$, then the elements $b_i, i\in I$ are pairwise α_{x_0}-disjoint.
5. $x_0=e$ if and only if $a\mathrm{T}_{x_0}x_0$ for all $a\in S$.

Proof. The properties 1., 2. and 3. are obvious.

4. We have that $a_i \mathrm{T}_{x_0} a_j$, $a_i \le b_i$ and $a_j \le b_j$, for every $i, j \in I$ with $i \ne j$. Therefore, because of 3., we obtain $b_i \mathrm{T}_{x_0} b_j$, for every $i, j \in I$ with $i \ne j$.

5. Let $x_0 = e$ and $a \in S$. Let $y \in S$ such that $y \ge a, x_0$. Then it is clear that $y = x_0$ and therefore, $a \mathrm{T}_{x_0} x_0$. *Conversely*, if $a \mathrm{T}_{x_0} x_0$ for every $a \in S$, then $x_0 \mathrm{T}_{x_0} x_0$. Thus for all $y \in S$, $y \ge x_0$ implies $y = x_0$, that is, $x_0 = e$.

It is obvious the following proposition.

Proposition 2.4. *Let S be a le-semigroup. The following properties hold true:*

1. if $a \Pi_{x_0} b$, then $b \Pi_{x_0} a$.
2. if $a \Pi_{x_0} b$ and $a \le b$, then $b = x_0$.
3. $x_0 = e$ if and only if $a \Pi_{x_0} x_0$ for all $a \in S$.

Theorem 2.5. *If S is a $\vee e$-semigroup with a unit i, then*

1. if $a \Pi_i b, c \ge i$ and $ac \ge b$ (or $ca \ge b$), then $c \ge b$.

2. if $a \Pi_i b$ and $c \le i$, then $a \vee bc = a \vee c$ (and $a \vee cb = a \vee c$).

If moreover, S is a le-semigroup, then the following properties hold true:

3. a) suppose that the elements $a_j \in S; j = 1,...,n, n \ge 2$ are pairwise commutative and i-prime with $\prod_{j=1}^{n} a_j = c$. If $a_j \ne i$ for every $j = 1,...,n$, then $a_j \ne c$ for every $j = 1,...,n$.

b) for $n = 2$ the converse of a) is true.

c) for $n > 2$ the converse of a) is not true.

4. under the same hypothesis as in 3. a) hypothesis for the elements $a_j \in S; j = 1,...,n, n \ge 2$ we have $\prod_{j=1}^{n} a_j = i$ if and only if $a_j = i$ for every $j = 1,...,n$.

5. let that the elements $a_j \in S : j = 1,...,n$ are pairwise i-prime with $\prod_{j=1}^{n} a_j = a$ and that there exist elements $b_\lambda \in S; \lambda = 1,...,m$ pairwise commutative and i-prime with $\prod_{\lambda=1}^{m} b_\lambda = a_\mu (1 \le \mu \le n)$.

Then the elements $a_1,...,a_{\mu-1}, b_1,...,b_m, a_{\mu+1},...,a_n$ are also pairwise i-prime and

$$\left(\prod_{j=1}^{\mu-1} a_j\right)\left(\prod_{\lambda=1}^{m} b_\lambda\right)\left(\prod_{j=\mu+1}^{n} a_j\right) = a$$

Proof. 1. Let $a \Pi_i b, c \ge i$ and $ac \ge b$, then

$$c = (a \vee b)c = ac \vee bc \ge b \vee b = b.$$

2. Indeed: $a \vee c = a \vee ((a \vee b)c) = a \vee ac \vee bc = a \vee bc$ (since $ac \le a$).

3. a) The elements $a_j, j = 1,...,n, n \ge 2$ are pairwise commutative and i-prime. Therefore:

$$a_1 \wedge a_2 \wedge ... \wedge a_n = a_1 a_2 ... a_n$$

If we suppose now that there exists t $(1 \le t \le n)$ with $a_t = c$, then $a_t = a_1 \wedge a_2 \wedge ... \wedge a_n$. Therefore, $a_j \ge a_t$ for every $j = 1,...,n$ with $j \ne t$. But for every $j = 1,...,n$ with $j \ne t$, $a_j \Pi_i a_t$, thus $a_j = i$ for every $j = 1,...,n$ with $j \ne t$ (Proposition 2.4). This is impossible.

b) It is obvious.

c) Let $a_j \in S, j=1,...,n, n>2$, where $a_1 \neq i, a_2 \neq i$ and $a_3 = ... = a_n = i$. By one side, since 3. a) is true, we have $a_1 \wedge a_2 \wedge ... \wedge a_n = c$ and therefore, $c \le a_1$. But $a_1 < i$ (since $a_1 \Pi_i a_j$ for every $j \neq 1$ and $a_1 \neq i$), thus $c \neq i$, that is, $c \neq a_3, ..., c \neq a_n$. By the other side, $c = a_1 a_2$, where $a_1 \neq i$ and $a_2 \neq i$. This and 3. a) imply $a_1 \neq c$ and $a_2 \neq c$. Therefore, $c \neq a_1, ..., c \neq a_n$ and this completes the proof of the property.

4. Let $\prod_{j=1}^{n} a_j = i$. Then $a_1 \wedge a_2 \wedge ... \wedge a_n = a_1 a_2 ... a_n = i$. Therefore, $i \le a_j$ for all $j = 1,...,n$. Since $a_j, j=1,...,n$ are pairwise i-prime, we have $a_j \le i$, for all $j=1,...,n$. Therefore, $a_j = i$ for all $j = 1,...,n$. The converse is obvious.

5. The elements $b_\lambda \in S, \lambda = 1,...,m$ are pairwise commutative and i-prime. Therefore, $\bigwedge_{\lambda=1}^{m} b_\lambda = \prod_{\lambda=1}^{m} b_\lambda = a_\mu$. Thus we have $a_j \Pi_i a_\mu, a_j \le a_j$ and $a_\mu \le b_1, ..., b_m$ for all $j=1,...,n$ with $j \neq n$. Therefore, by Proposition 2.4, we have $a_j \Pi_i b_1, ..., b_m$ for all $j = 1,...,n$ with $j \neq \mu$ which implies the property.

Remark 1. (a) The property 4. of the above theorem implies that 3. makes sense only for $c \neq i$.
(b) The property 1. remains true in case S is a $\vee e$-semigroup.

3. ALGEBRA OF POLAR SUBSETS OF A *LE*-SEMIGROUP

Let S be a le-semigroup, $x_0 \in S$ and $A \neq \varnothing$ a subset of S. Then the following set defined by

$$A_{x_0}^{\Pi} = \{x \in S : x \Pi_{x_0} a, \forall a \in A\}$$

is called Π_{x_0}-*polar* of A. If $A_{x_0}^{\Pi} \neq \varnothing$, then the set $(A_{x_0}^{\Pi})_{x_0}^{\Pi}$ is called Π_{x_0}-*bipolar* of A and we denote it $A_{x_0}^{\Pi\ \Pi}$.

Remark 1. The Π_{x_0}-*polar* and Π_{x_0}-*bipolar* of A are different in general.

Proposition 3.1. *Let S be a le-semigroup, $x_0 \in S$ and $A \neq \varnothing$ a subset of S. Then $A_{x_0}^{\Pi} \neq \varnothing$ if and only if x_0 is the greatest element of $A_{x_0}^{\Pi}$*. If $A_{x_0}^{\Pi} \neq \varnothing$, then

a) $x_0 \ge a$ for every $a \in A$ and b) $x_0 \ge x$ for every $x \in A_{x_0}^{\Pi}$.

But a) implies $x_0 \vee a = x_0$ for every $a \in A$, that is, $x_0 \in A_{x_0}^{\Pi}$. Therefore, x_0 is the greatest element of $A_{x_0}^{\Pi}$. The converse is obvious.

Proposition 3.2. *Let S be a le-semigroup and $x_0 \in S$. The following statements are equivalent*

1. $x_0 = e$.

2. for every $\varnothing \neq A \in \mathcal{P}(S), x_0 \in A_{x_0}^{\Pi}$, where $\mathcal{P}(S)$ denotes the set of all subsets of S.

3. for every $\varnothing \neq A \in \mathcal{P}(S), A_{x_0}^{\Pi} \neq \varnothing$.

Proof. $1. \Rightarrow 2.$ Indeed: if $x_0 = e$, then $x_0 \Pi_{x_0} a$, for all $a \in A$ and for all $\varnothing \neq A \in \mathcal{P}(S)$. This implies the 2.

$2. \Rightarrow 3.$ It is obvious.

$3. \Rightarrow 1.$ Indeed: if 3. is true, then $\{a\}_{x_0}^{\Pi} \neq \varnothing$ for all $a \in S$. Therefore, $x_0 \in \{a\}_{x_0}^{\Pi}$ for all $a \in S$ (Proposition 3.1). Thus $x_0 \geq a$ for all $a \in S$, that is, $x_0 = e$.

Let S be a *le*-semigroup. If $A \subseteq S$, the *polar* of A, denoted by A^*, is the set defined by

$$A^* = \{x \in S : x \vee a = e, \forall a \in A\} = A_e^{\Pi}$$

The set A^{**} is called the *bipolar* of A.

Proposition 3.3. *Let S be a le-semigroup and $\varnothing \neq A, B \subseteq S$. Then the following hold true:*

1. if $A \subseteq B$, then $A^* \supseteq B^*$.
2. $A \subseteq A^{**}$.
3. $A^* = A^{***}$.
4. $A^* \cap A^{**} = \{e\}$.

Proof. 1. It is obvious.

2. If $a \in A$ and $x \in A^*$, then $a \vee x = e$. Therefore $a \in A^{**}$.

3. By 2., we have $A^* \subseteq A^{***}$. By 1., since $A \subseteq A^{**}$, we obtain $A^* \supseteq A^{***}$.

4. We have $A^* \cap A^{**} \subseteq \{e\}$. In fact: if $y \in A^* \cap A^{**}$, then $y \vee a = e$, for all $a \in A^*$, but then $y \vee y = e$, thus $y = e$. By Proposition 3.2, we obtain $e \in A^* \cap A^{**}$.

An immediate corollary of Proposition 3.2 and 3.3 is the following proposition:

Corollary 3.4. *Let S be a le-semigroup and $x_0 \in S$. x_0 is the greatest element (= e) of S if and only if for every $\varnothing \neq A \subseteq S$, $A_{x_0}^{\Pi} \cap A_{x_0}^{\Pi\ \Pi} = \{x_0\}$.* In following, for a *le*-semigroup S and $x_0 \in S$ we will denote $\mathcal{A}_{x_0}^{\Pi} = \{A_{x_0}^{\Pi} : \varnothing \neq A \subseteq S\}$ and $\mathcal{B} = \{A \subseteq S : A = A^{**}\}$.

Corollary 3.5. *Let S be a le-semigroup. Then $\mathcal{B} = \mathcal{A}_e^{\Pi}$.*

Proof. If $A^* \in \mathcal{A}_e^{\Pi}$, then $A^* = A^{***}$ (Proposition 3.3(3.)). Therefore $A^* \in \mathcal{B}$. Conversely, if $A \in \mathcal{B}$, then $A = A^{**}$. Therefore $A \in \mathcal{A}_e^{\Pi}$.

Corollary 3.6. *Let S be a le-semigroup and $\varnothing \neq A \subseteq S$. Then the bipolar A^{**} of A is the smallest element of $\mathcal{B}$ containing A.*

Proof. Corollary 3.5 implies that A^{**} is an element of $\mathcal{B}$. Proposition 3.3(2.) implies that $A \in \mathcal{B}$. Let assume that there exists $A' \in \mathcal{B}$, where $A \subseteq A' \subseteq A^{**}$. Then, Proposition 3.3(1., 3.) implies that $A^* \supseteq (A')^* \supseteq A^{***} = A^*$. Therefore $(A')^* = A^*$. Since $A' \in \mathcal{B}$, then we have $A' = A^{**}$.

Proposition 3.7. *Let S be a le-semigroup and $A_i \in \mathcal{B}, i \in I$. Then $\bigcap_{i \in I} A_i \in \mathcal{B}$.*

Proof. For all the families $\{A_i : i \in I\}$ of subsets of S we have $\left(\bigcup_{i \in I} A_i\right)^* = \bigcap_{i \in I} (A_i)^* \ldots \qquad (A)$

Indeed: let $x \in \left(\bigcup_{i \in I} A_i\right)^*$. This is equivalent with $x \vee a = e, \forall a \in A_i, \forall i \in I$. That is, $x \in (A_i)^*, \forall i \in I$. So, $x \in \bigcap_{i \in I} (A_i)^*$. By (A), we have:

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I} \left((A_i)^*\right)^* = \left(\bigcap_{i \in I} (A_i)^*\right)^*$$
$$= \left(\bigcup_{i \in I} (A_i)^*\right)^{***} \quad (\textit{Proposition } 3.3(3.)$$
$$= \left(\bigcap_{i \in I} (A_i)^{**}\right)^{**} = \left(\bigcap_{i \in I} A_i\right)^{**}.$$

Corollary 3.8. *Let* S *be a le-semigroup and* $\{A_i : i \in I\}$ *a family of subsets of* S. *Then:*

1. $\left(\bigcup_{i\in I} A_i\right)^* = \left(\bigcup_{i\in I} (A_i)^{**}\right)^*$.

2. if, furthermore $A_i \in \mathcal{B}$ for all $i \in I$, then $\left(\bigcap_{i\in I} A_i\right)^* = \left(\bigcup_{i\in I} (A_i)^*\right)^{**}$.

Proof. (1).

$$\left(\bigcup_{i\in I} (A_i)^{**}\right)^* = \bigcap_{i\in I} (A_i)^{***} \text{(Proposition 3.7(A))}$$
$$= \bigcap_{i\in I} (A_i)^* \text{(Proposition 3.3(3.)}$$
$$= \left(\bigcup_{i\in I} A_i\right)^* \text{(Proposition 3.7(A))}.$$

(2). In fact, $\bigcap_{i\in I} A_i = \bigcap_{i\in I} (A_i)^{**} = \left(\bigcup_{i\in I} (A_i)^*\right)^*$ (Proposition 3.7(A)). Therefore $\left(\bigcap_{i\in I} A_i\right)^* = \left(\bigcup_{i\in I} (A_i)^*\right)^{**}$.

Definition 3.9. *A nonempty subset* A *of an ordered semigroup* S *will be called semi-filter in* S *if and only if* $z \in A$ *and* $x \in S$ *with* $x \geq z$, *implies* $x \in A$.

Proposition 3.10. *Let* S *be a le-semigroup. Then every* $A \in \mathcal{B}$ *is a semi-filter in* S.

Proof. If $A \in \mathcal{B}, a \in A$ and $b \in S$ with $b \geq a$, then $b \vee x \geq a \vee x = e$ for every $x \in A^*$. Since e is the greatest element of S, then $b \vee x = e$ for every $x \in A^*$, that is, $b \in A^{**} = A$.

Proposition 3.11. *Let* S *be a le-semigroup and* A_1, A_2 *are semi-filter in* S. *Then:*

$$(A_1 \cap A_2)^{**} = (A_1)^{**} \cap (A_2)^{**}.$$

Proof. From Proposition 3.3(3.), it follows:

$$(A_1 \cap A_2)^{**} \subseteq (A_1)^{**} \cap (A_2)^{**}.$$

Now it is enough to prove that: $x \in (A_1)^{**} \cap (A_2)^{**}$ and $y \in (A_1 \cap A_2)^*$ imply $x \vee y = e$.

From Proposition 3.2 and Proposition 3.10, it follows that $(A_1)^{**}, (A_2)^{**}$ and $(A_1 \cap A_2)^*$ are semi-filter in S. We have the following:

$$x \vee y \geq x \in (A_1)^{**} \cap (A_2)^{**}, \text{that is, } x \vee y \in (A_1)^{**} \cap (A_2)^{**},$$
$$x \vee y \geq y \in (A_1 \cap A_2)^*, \text{that is, } x \vee y \in (A_1 \cap A_2)^*,$$

then $x \vee y \in (A_1)^{**} \cap (A_2)^{**} \cap (A_1 \cap A_2)^*$ . . . (1).

Let now $u \in A_1$ and $v \in A_2$. Since $A_i, i = 1, 2$ are semi-filter in S and $u \vee v \geq u, v$, it follows that: $u \vee v \in A_1 \cap A_2$, then $u \vee v \in (A_1 \cap A_2)^{**}$ (Proposition 3.3(2.) . . . (2).

From (1) and (2), we have $x \vee y \vee u \vee v = e$ (Proposition 3.3(4.), that is, $x \vee y \vee u \in (A_2)^*$. Also,

$$x \vee y \vee u \geq x \vee y \in (A_2)^{**}, \text{ that is, } x \vee y \vee u \in (A_2)^{**}.$$

From Proposition 3.3(4.), we have $x \vee y \vee u = e$. Also, $x \vee y \in (A_1)^*$ and $x \vee y \in (A_1)^{**}$. Then $x \vee y = e$ (Proposition 3.3(4.).

Theorem 3.12. *Let* S *be a* le*-semigroup. Then the set* $\mathcal{B}$ *partially ordered by set inclusion is a complete Boolean algebra.*

Proof. It is clear that the set $\mathcal{B}$, partially ordered by set inclusion, is a partially ordered semigroup. Let

now $A_i \in \mathcal{B}, i \in I$. From Proposition 3.7, it follows that $\bigcap_{i\in I} A_i \in \mathcal{B}$, then $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$ (since if $S \in \mathcal{B}$ with $S \subseteq A_i, \forall i \in I$, then $S \subseteq \bigcap_{i\in I} A_i$). Also, from Proposition 3.3, it follows that $\bigvee_{i\in I} A_i = \left(\bigcup_{i\in I} A_i\right)^{**}$.
Therefore, $\mathcal{B}$ is a complete lattice.
From Proposition 3.3(2.), it follows that $S \in \mathcal{B}$, that is, S is the greatest element of $\mathcal{B}$. Since

(1) $e \in A$ for every $A \in \mathcal{B}$ and
(2) $e \in \mathcal{B}$,

then $\{e\}$ is the least element of $\mathcal{B}$. Indeed: by one side, $e \in S^*$ (Proposition 3.2). By the other side, if $e \in S^*$, then $x \vee y = e$ for all $y \in S$. It follows that for $y = x$ we have $x = e$. Therefore $S^* = \{e\}$. From this and Proposition 3.2, it follows that $\{e\} \in \mathcal{B}$. (1) can be taken as an immediate consequence of Proposition 3.2.

The le-semigroup $\mathcal{B}$ is complemented as lattice. Indeed: if $A \in \mathcal{B}$, then A^* is an element of $\mathcal{B}$ (Proposition 3.2), such that:

$$A \wedge A^* = A \cap A^* = A^{**} \cap A^* = \{e\} \text{ (Proposition 3.3(4.) and}$$
$$A \vee A^* = (A \cup A^*)^{**} = (A^* \cap A^{**})^* \text{ (Proposition 3.7(A))}$$
$$= \{e\}^* = S \text{ (since } e \text{ is the greatest element of } S,$$
$$\text{we have for all } x \in S, x \in \{e\}^*).$$

It remains to show that the le-semigroup $\mathcal{B}$ is distributive. Indeed: let $A_1, A_2, A_3 \in \mathcal{B}$, then

$$A_1 \vee (A_2 \vee A_3) = A_1 \cap (A_2 \cap A_3)^{**} = (A_1)^{**} \cap (A_2 \cup A_3)^{**}$$
$$= (A_1 \cap (A_2 \cup A_3)^{**} \text{ (by Proposition 3.10 and 3.11)}$$
$$= \left((A_1 \cap A_2) \cup (A_1 \cap A_3)\right)^{**} = (A_1 \wedge A_2) \vee (A_1 \wedge A_3).$$

Next, consider the case where S is a distributive le-semigroup.

Proposition 3.13. *Let S a le-semigroup and i the identity element of S. Then every element of $\mathcal{A}_i^{\Pi}$ is a convex le-subsemigroup of S.*

Proof. Let $A_i^{\Pi} \in \mathcal{A}_i^{\Pi}, x_1, x_2 \in A_i^{\Pi}$ and $a \in A$, then $x_1 \Pi_i a$ and $x_2 \Pi_i a$. But then $x_1 x_2 \Pi_i a, x_1 \wedge x_2 \Pi_i a$ and $x_1 \vee x_2 \Pi_i a$. Therefore, $x_1, x_2 \in A_i^{\Pi}$ implies $x_1 x_2, x_1 \wedge x_2, x_1 \vee x_2 \in A_i^{\Pi}$. Thus A_i^{Π} is a le-subsemigroup of S. It remains to show that A_i^{Π} is convex. In fact, if $x_1, x_2 \in A_i^{\Pi}$ and $y \in S$ with $x_1 \le y \le x_2$, then for all $a \in A$, $i = x_1 \vee a \le y \vee a \le x_2 \vee a = i$. Thus for all $a \in A$, $y \vee a = i$. Therefore $y \in A_i^{\Pi}$.

Remark 2. For every le-semigroup S and $x_0 \in S$, every element of $\mathcal{A}_{x_0}^{\Pi}$ is in general convex.

The following theorem is an immediately corollary of Proposition 3.13, Theorem 3.12 and Corollary 3.5.

Theorem 3.14. *Let S be a distributive le-semigroup with e as identity element. Then the set $\mathcal{B}$ partially ordered by set inclusion is a complete Boolean algebra and every element of the algebra $\mathcal{B}$ is a convex le-subsemigroup of S.*

4. THE EMBEDDING OF $\mathcal{S}$ IN $\mathcal{B}$

Let S be a distributive le-semigroup. For $a, b \in S$, we put $a \approx b$ if and only if $a^* = b^*$. It is clear that $\approx$ is an equivalence relation in S. Let $\mathcal{S}$ be the set of corresponding equivalence classes and $a^{\wedge}$ the class containing a.

Proposition 4.1. *Let* S *be a distributive* le*-semigroup. Then* $\mathcal{S}$ *is also a distributive* le*-semigroup.*
Proof. We define in $\mathcal{S}$ an order relation as follows:

$$a^{\wedge} \leq b^{\wedge} \text{ if and only if } a^{*} \subseteq b^{*}.$$

It can be easily seen that this definition is independent from the representative elements a, b of $a^{\wedge}, b^{\wedge}$. The mapping $S \ni a \to a^{\wedge} \in \mathcal{S}$ is isotone. Indeed: if $a, b \in S, a \leq b$ and $x \in a^{*}$, then $e = x \vee a \leq x \vee b$. Therefore $e = x \vee b$, that is, $x \in b^{*}$. This implies that $(a \vee b)^{\wedge} \geq a^{\wedge}, b^{\wedge}$ and $(a \wedge b)^{\wedge} \leq a^{\wedge}, b^{\wedge}$. If $c^{\wedge} \in \mathcal{S}$ with $c^{\wedge} \geq a^{\wedge}, b^{\wedge}$ and $x \in (a \vee b)^{*}$, then $x \vee (a \vee b) = e$. This implies $b \vee x \in a^{*} \subseteq c^{*}$, so $(b \vee x) \vee c = e$. But then $c \vee x \in b^{*} \subseteq c^{*}$. Therefore $c \vee x = e$, that is, $x \in c^{*}$ and if $c^{\wedge} \in \mathcal{S}$ with $c^{\wedge} \leq a^{\wedge}, b^{\wedge}$ and $x \in c^{*}$, then, since $x \in a^{*} \cap b^{*}$, we have $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) = e$. So, $x \in (a \wedge b)^{*}$. Thus, $\mathcal{S}$ is a le-semigroup with $(a \vee b)^{\wedge} = a^{\wedge} \vee b^{\wedge}$ and $(a \wedge b)^{\wedge} = a^{\wedge} \wedge b^{\wedge}$. Also, $\mathcal{S}$ is distributive. Indeed:

$$\begin{aligned} a^{\wedge} \wedge (b^{\wedge} \vee c^{\wedge}) &= a^{\wedge} \wedge (b \vee c)^{\wedge} = (a \wedge (b \vee c))^{\wedge} \\ &= ((a \wedge b) \vee (a \wedge c))^{\wedge} = (a \wedge b)^{\wedge} \vee (a \wedge c)^{\wedge} \\ &= (a^{\wedge} \wedge b^{\wedge}) \vee (a^{\wedge} \wedge c^{\wedge}). \end{aligned}$$

Proposition 4.2. *Let* S *be a distributive* le*-semigroup. Then the following hold in* $\mathcal{B}$:

1. $(a \wedge b)^{*} = a^{*} \wedge b^{*}$.
2. $(a \wedge b)^{**} = a^{**} \vee b^{**}$.
3. $(a \vee b)^{**} = a^{**} \wedge b^{**}$.
4. $(a \vee b)^{*} = a^{*} \vee b^{*}$.

Proof. 1. We have : $x \in (a \wedge b)^{*} \Leftrightarrow x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) = e \Leftrightarrow e \leq x \vee a$ and $e \leq x \vee b \Leftrightarrow e = x \vee a = x \vee b \Leftrightarrow x \in a^{*} \wedge b^{*}$.

2. We have $(a \wedge b)^{*} = a^{*} \cap b^{*} = (\{a\} \cup \{b\})^{*} = \{a, b\}^{*}$. But $\{a, b\} \subseteq a^{**} \cup b^{**}$. So, $\{a, b\}^{*} \supseteq (a^{**} \cup b^{**})^{*}$. By the other side, if $x \in \{a, b\}^{*}$, then $x \Pi_{e} a, b$. So, $x \in a^{*} = a^{***}$ and $x \in b^{*} = b^{***}$, then for every $y \in a^{**} \cup b^{**}$, $x \Pi_{e} y$. Therefore $x \in (a^{**} \cup b^{**})^{*}$. Hence we have

$$\begin{aligned} (a \wedge b)^{**} &= (a^{**} \cup b^{**})^{**} \\ &= a^{**} \vee b^{**}. \end{aligned}$$

3. By the Proposition 3.3(2.), 3.10 and Corollary 3.5, we have $a \vee b \geq a \in a^{**}$, so $a \vee b \in a^{**}$. In similar, we have $a \vee b \geq b \in b^{**}$, so $a \vee b \in b^{**}$. Then $a \vee b \in a^{**} \cap b^{**} \in \mathcal{B}$, thus $(a \vee b)^{**} \subseteq (a^{**} \cap b^{**})^{**} = a^{**} \cap b^{**}$.

Let $x \in a^{**} \cap b^{**}$ and $y \in (a \vee b)^{*}$, then $y \vee a \vee b = e$. Thus $x \vee y \vee a \vee b = e$. Therefore $x \vee y \vee a \in b^{*}$. But $x \vee y \vee a \geq x \in b^{**}$, so $x \vee y \vee a \in b^{**}$. Also, $x \vee y \vee a = e$ and so $x \vee y \in a^{*}$. But $x \vee y \geq x \in a^{**}$ and so $x \vee y \in a^{**}$. Therefore $x \vee y = e$, that is, $x \Pi_{e} y$.

4. We have

$$\begin{aligned} (a \vee b)^{*} &= (a \vee b)^{***} = (a^{**} \wedge b^{**})^{*} (by(3)) \\ &= (a^{*} \cup b^{*})^{**} = a^{*} \vee b^{*}. \end{aligned}$$

Corollary 4.3. *Let* S *be a distributive* le*-semigroup. Then every element of* $\mathcal{S}$ *is a convex* le*-subsemigroup of* S.
Proof. Let $a^{\wedge} \in \mathcal{S}, b, c \in a^{\wedge}$ and $x \in S$ with $b \leq x \leq c$. Then $a^{*} = b^{*} \subseteq x^{*} \subseteq c^{*} = a^{*}$. Therefore $x^{*} = a^{*}$, that is, $x \in a^{\wedge}$. Thus $a^{\wedge}$ is a convex set. More, $a^{\wedge}$ is a le-subsemigroup of S. Indeed: let $b, c \in a^{\wedge}$. Then $b^{*} = a^{*} = c^{*}$. By Proposition 4.2(1.,4.), we have $(b \wedge c)^{*} = a^{*} \wedge a^{*} = a^{*}$ and

$(b \vee c)^* = a^* \vee a^* = (a^*)^{**}$. Therefore, $b \wedge c \in a^\wedge$ and $b \vee c \in a^\wedge$.

It is clear that $a^\wedge \neq b^\wedge$ implies $a^* \neq b^*$ (since $a^* = b^*$, then $a \approx b$). Then by Proposition 4.2 (1., 4.), it follows that the mapping $\mathcal{S} \ni a^\wedge \to a^* \in \mathcal{B}$ is an isomorphism.

5. ALGEBRA OF POLAR SUBSETS OF A *POE*-SEMIGROUP

Let S be a poe-semigroup, $x_0 \in S$ and $A \neq \varnothing$ a subset of S. Then the following set defined by

$$A_{x_0}^{\mathrm{T}} = \{x \in S : x \mathrm{T}_{x_0} a, \forall a \in A\}$$

is called T_{x_0} -*polar* of A. If $A_{x_0}^{\mathrm{T}} \neq \varnothing$, then the set $(A_{x_0}^{\mathrm{T}})_{x_0}^{\mathrm{T}}$ is called T_{x_0} -*bipolar* of A and we denote it $A_{x_0}^{\mathrm{T}\ \mathrm{T}}$.

Let we denote $\mathcal{B}_{x_0}^{\mathrm{T}} = \left\{A \subseteq S : A = A_{x_0}^{\mathrm{T}\ \mathrm{T}}\right\}$.

The following proposition can be proved in similar way with Proposition 3.3.

Proposition 5.1. *Let S be a poe-semigroup, $x_0 \in S$ and $A, B \subseteq S$. The following hold true:*

1. if $A \subseteq B$, then $A_{x_0}^{\mathrm{T}} \supseteq B_{x_0}^{\mathrm{T}}$.
2. $A \subseteq A_{x_0}^{\mathrm{T}\ \mathrm{T}}$.
3. $A_{x_0}^{\mathrm{T}} = A_{x_0}^{\mathrm{T}\ \mathrm{T}\ \mathrm{T}}$.
4. $A_{x_0}^{\mathrm{T}} \cap A_{x_0}^{\mathrm{T}\ \mathrm{T}} \subseteq \{x_0\}$.

Proposition 5.2. *Let S be a poe-semigroup, $x_0 \in S$ and $A \subseteq S$. Then $A_{x_0}^{\mathrm{T}} \cap A_{x_0}^{\mathrm{T}\ \mathrm{T}} = \{x_0\}$ if and only if $x_0 = e$.*

Proof. Let $A_{x_0}^{\mathrm{T}} \cap A_{x_0}^{\mathrm{T}\ \mathrm{T}} = \{x_0\}$, then $x_0 \in A_{x_0}^{\mathrm{T}\ \mathrm{T}}$. Therefore, for all $a \in A_{x_0}^{\mathrm{T}}$, $x_0 \mathrm{T}_{x_0} a$. Thus $x_0 \mathrm{T}_{x_0} x_0$, so $x_0 = e$. Now, if $x_0 = e$, then for all $A \subseteq S$, $x_0 \in A_{x_0}^{\mathrm{T}}$, therefore, the converse is true.

Proposition 5.3. *Let S be a poe-semigroup, $x_0 \in S$ and $A \subseteq S$. Then $A_{x_0}^{\mathrm{T}}$ is semi-filter in S.*

Proof. Let $z \in A_{x_0}^{\mathrm{T}}$ and $x \in S$ with $x \geq z$. If $a \in A$ and $b \geq x, a$, then, since $b \geq z, a$, where $a \in A$ and $z \in A_{x_0}^{\mathrm{T}}$, it follows that $b = x_0$, so $x \in A_{x_0}^{\mathrm{T}}$.

Proposition 5.4. *Let S be a poe-semigroup, $x_0 \in S$ and $A_i, i = 1, 2$ semi-filter in S. Then*

$$(A_1 \cap A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}} = (A_1)_{x_0}^{\mathrm{T}\ \mathrm{T}} \cap (A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}}.$$

Proof. Proposition 5.1 implies

$$(A_1 \cap A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}} \subseteq (A_1)_{x_0}^{\mathrm{T}\ \mathrm{T}} \cap (A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}}.$$

Let now $x \in (A_1)_{x_0}^{\mathrm{T}\ \mathrm{T}} \cap (A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}}, y \in (A_1 \cap A_2)_{x_0}^{\mathrm{T}}$ and $a \in S$ with $a \geq x, y$. We have to show that $a = x_0$. First, we have $a \in (A_1)_{x_0}^{\mathrm{T}\ \mathrm{T}} \cap (A_2)_{x_0}^{\mathrm{T}\ \mathrm{T}} \cap (A_1 \cap A_2)_{x_0}^{\mathrm{T}}$ (by Proposition 5.3). Let now

$u \in A_1, v \in A_2$. If $z, t \in S$ with $z \geq a, u$ and $t \geq z, v$, then, since $a \in (A_1 \cap A_2)_{x_0}^{\top}$, it follows that $t \in (A_1 \cap A_2)_{x_0}^{\top}$ (by Proposition 5.3). But, $t \geq z \geq u$, so $t \in A_1$ and $t \geq u$, so $t \in A_2$. Then $t \in A_1 \cap A_2 \subseteq (A_1 \cap A_2)_{x_0}^{\top\ \top}$. Therefore $t = x_0$.

Thus, we showed that for all $v \in A_2$, $z \top_{x_0} v$, that is, $z \in (A_2)_{x_0}^{\top}$. By the other side, $a \in (A_2)_{x_0}^{\top\ \top}$, so $z \in (A_2) \top\ \top_{x_0}$. Thus $z = x_0$. Therefore, we have for all $u \in A_1$, $a \top_{x_0} u$, that is, $a \in (A_1)_{x_0}^{\top}$. But $a \in (A_2)_{x_0}^{\top\ \top}$, thus $a = x_0$.

The following proposition can be proved in similar way as in Proposition 3.3 and 3.7.

Proposition 5.5. *Let S be a poe-semigroup and $x_0 \in S$. Then*

1. if $A \subseteq S$, then $A_{x_0}^{\top\ \top}$ is the least element in $\mathcal{B}_{x_0}^{\top}$ containing A.
2. if $A_i \in \mathcal{B}_{x_0}^{\top} (i \in I)$, then $\bigcap_{i \in I} A_i \in \mathcal{B}_{x_0}^{\top}$.

Proposition 5.6. *Let S be a poe-semigroup. Then every $A \in \mathcal{B}_{x_0}^{\top}$ is a semi-filter in S.*

Proof. Let $A \in \mathcal{B}_{x_0}^{\top}, a \in A, b \in S$ with $b \geq a, z \in A_{x_0}^{\top}$ and $t \in S$ with $t \geq b, z$. Then, since $t \geq a, z$, where $a \in A$ and $z \in A_{x_0}^{\top}$, it follows that $t = x_0$. Thus, for all $z \in A_{x_0}^{\top}, b \top_{x_0} z$. Therefore, $b \in A_{x_0}^{\top\ \top} = A$ and this completes the proof.

In similar way as in the proof of Theorem 3.12 it can be proved the following theorem.

Theorem 6 *Let S be a poe-semigroup. Then $\mathcal{B}_{x_0}^{\top}$ partially ordered by set inclusion is a complete Boolean algebra.*

REFERENCES

[1] Bernau S.J. (1965) *Unique representation of archimedean lattice groups and normal archimedean lattice rings*. London Math. Soc. (3) 15, 559-631.

[2] Birkhoff G. (1967) *Lattice Theory*, Vol. 25, Amer. Math. Soc. Col. Publ.

[3] Choudhury A.C. (1957) *The doubly distributive m-lattice*. Bull. Calcutta Math. Soc. 49, 71-74.

[4] Fuchs L. (1963) *Partially ordered algebraic systems*, Pergamon Press.

[5] Hila K. (2008) *On normal elements and normal closure in* $\vee e - \Gamma$*-semigroup*s. Algebras, Groups and Geometries, 25, No. 1, 93-108.

[6] Kappos D.A., Kehayopulu N. (1971) *Some remarks on the representation of lattice ordered groups*. Mathematica Balkanica 1, 142-143.

[7] Kehayopulu N. (1971) *m-Lattices and the algebra of polar subsets of a lattice*. Bull. Soc. Math. Grece 12, No.2, 225-282.

[8] MacNeille H. (1937) *Partially ordered sets*. Trans. Amer. Math. Soc. 42, 416-460.

[9] MacNeille H. (1939) *Extension of a distributive lattice to a Boolean ring*. Bull. Amer. Math. Soc. 45, 452-455.