ISSN 2073-2244
www.alb-shkenca.org

AKTET
Journal of Institute Alb-Shkenca
Revistë Shkencore e Institutit Alb-Shkenca

CONNECTION BETWEEN DIFFERENTIABILITY AND ONE-SIDED DIFFERENTIABILITY

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ABSTRACT

In this paper, we give the generalization of a theorem having to do with Lebesgue integration of derivative of a function. We demonstrate that correctness of this theorem remains by replacing "the derivative" with "the right-hand derivative". The main purpose of this paper is the new way of studying of the existing relations between "the derivative" of a continuous function on the one side, and "the right-hand derivative" or "right-hand derivative numbers" on the other side.

Key words: derivative, right-hand derivative, right-hand derivative number, Lebesgue integral.

PËRMBLEDHJE

Në këtë artikull jepet përgjithësimi i një teoreme që ka të bëjë me integrimin sipas Lebegut të derivatit të një funksioni. Kemi provuar se vërtetësia e kësaj teoreme ruhet nëse zëvendësojmë në te “derivatin” me “derivatin e djathtë”. Qëllimi kryesor i artikullit është studimi në një mënyrë të re i lidhjeve që ekzistojnë ndërmjet “derivatit” të një funksioni të vazhdueshëm nga njëra anë, dhe “derivtatit të djathtë” apo “numrave derivativë të djathtë” të tij nga ana tjetër.

1. INTRODUCTION

The problems of relations between differentiability and one-sided differentiability of a function are well known in literature. The purpose of this paper is to present an analysis for one of these problems that has to do with investigation of relations that exist between the derivative of a continuous function and the right-hand derivative or right-hand derivative numbers.

Aside from analyzing of these relations, we also found the conditions in which the right-hand derivative of a function is equal with its derivative.

Let $f:[a,b]\to B$ be a function of the real variable x, where B is a Banach space (a complete normed space), then the right-hand derivative of the function f, denoted by $f_+^{'}$, is defined to be,

$$f_+^{'}(x)=\lim_{\substack{h\to 0\\(h>0)}}\frac{f(x+h)-f(x)}{h}$$

To achieve the aim mentioned above we will use the next 2 lemmas:

Lemma 1. If $f:[a,b]\to B$ is a continuous function on interval $[a,b]$, and the right-hand derivative $f_+^{'}$ exists and it is bounded on $[a,b)$, then there is a constant $K\geq 0$, that for every $x\in[a,b)$ satisfies $\left\|f_+^{'}(x)\right\|\leq K$ as,

$$\left\|f(b)-f(a)\right\|\leq K(b-a). \qquad (1)$$

The proof for Lemma (1) is given in [2]. Inequality (1) can be also written as:

$$\left\|f(b)-f(a)\right\|\leq (b-a)\sup_{t\in[a,b)}\left\|f_+^{'}(t)\right\| \qquad (2)$$

Lemma 2. Let $f:[a,b]\to R$ be a continuous function on interval $[a,b]$. Let us suppose that for any point $x\in[a,b)$ the right-hand derivative $f_+^{'}(x)$ exists. Then, the following inequality is true for some c and d in (a,b).

$$f_+^{'}(c)\leq\frac{f(b)-f(a)}{b-a}\leq f_+^{'}(d) \qquad (3)$$

A proof of this Lemma can be found in [4] or [8].

2. GENERALIZATION OF SOME RESULTS ABOUT FUNCTIONS THAT HAVE BOUNDED RIGHT-HAND DERIVATIVES

Theorem 1. If the function $f:[a,b]\to B$ is continuous on the finite interval $[a,b]$ and has continuous right-hand derivative on $[a,b)$, than function f has a continuous derivative on $[a,b)$.

Proof. Consider a fixed point $x_0\in[a,b)$ and any point $x\in[a,b)$. Presenting the inequality (2) on interval $[x_0,x]$ $(x>x_0)$ for the function and performing transformations we obtain,

$$x \text{ a } F(x)=f(x)-f'_+(x_0)(x-x_0) \tag{4}$$

$$\|F(x)-F(x_0)\|\le(x-x_0)\sup_{t\in[x_0,x]}\|F'_+(t)\| \tag{5}$$

$$\begin{aligned}&\|f(x)-f(x_0)-f'_+(x_0)(x-x_0)\|\le\\&\le(x-x_0)\sup_{t\in[x_0,x]}\|f'_+(t)-f'_+(x_0)\|\end{aligned} \tag{6}$$

Dividing by $(x-x_0)$,

$$\left\|\frac{f(x)-f(x_0)}{x-x_0}-f'_+(x_0)\right\|\le\sup_{t\in[x_0,x]}\|f'_+(t)-f'_+(x_0)\| \tag{7}$$

It is obvious that the inequality (7) maintains the same form even if we write inequality (2) for the function F on interval $[x_0,x]$ $(x>x_0)$. Passing to the limit on the both sides of (7) when $x\to x_0$ $(x\ne x_0)$, and take into consideration that function f'_+ is continuous at point x_0, we obtain the following,

$$0\le\left\|\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}-f'_+(x_0)\right\|\le0 \tag{8}$$

Thus, we conclude

$$0\le\|f'(x_0)-f'_+(x_0)\|\le0 \quad\text{or}$$

$$f'(x_0)=f'_+(x_0) \tag{9}$$

Since x_0 is an arbitrary point, the derivative $f'(x)$ exists for every $x\in[a,b)$, assuming that f' is continuous on $[a,b)$ like f'_+.

Corollary. For continuous functions on a closed interval, the existence and continuity of the right-hand derivative f'_+ is a necessary and sufficient condition for the existence and the continuity of the derivative f'.

Theorem 2. Let, $f:[a,b]\to R$ be a continuous function on interval $[a,b]$, which has right-hand derivative $f'_+(t)$ at every point $t\in[a,b)$. If the function f'_+ is bounded on $[a,b)$, then f'_+ is Lebesgue integrable on every closed interval $[a,x]\subset[a,b)$ and the following relationship is true.

$$(\mathrm{L})\int_a^x f'_+(t)dt=f(x)-f(a) \tag{10}$$

Proof. Let us apply lemma 2 to the function f on an arbitrary interval $[u,v]\subset[a,x]$. This implies that the inequalities (11) is true for some c, d in (u,v).

$$f'_+(c)(v-u)\le f(v)-f(u)\le f'_+(d)(v-u) \tag{11}$$

According to the hypothesis, the right-hand derivative f'_+ is bounded on $[a,b)$. This implies that f'_+ is bounded on $[a,x]$, and there exists a constant $M>0$ such that $-M\le f'_+(t)\le M$ for all $t\in[a,x]$. From (11) we obtain the followings for any interval $[u,v]\subset[a,x]$.

$$-M(v-u)\le f(v)-f(u)\le M(v-u) \quad\text{or}$$

$$|f(v)-f(u)|\le M(v-u)$$

This implies that the function f is absolutely continuous on $[a,x]$. According to Lebesgue theorem (see [5] p.334-335) the derivative f' is integrable on $[a,x]$ and,

$$(\mathrm{L})\int_a^x f'(t)dt=f(x)-f(a) \tag{12}$$

The formula (10) derives from (12) substituting $f'_+=f'$.

Corollary 1. There is no function from $C_{[a,b)}$ whose right-hand derivative on $[a,b)$ equals the Dirichlè function

$$\chi(x)=\begin{cases}1, & \text{if } x \text{ is a rational number};\\0, & \text{if } x \text{ is a irrational number}.\end{cases}$$

Let us suppose that there exists a function $g \in C_{[a,b)}$ such that

$$\forall x \in [a,b),\ g_+^{'}(x)=\chi(x).$$

Since conditions of the theorem 1 are satisfied, formula (10) can be written as,

$$\forall x \in [a,b);\ g(x)=g(a)+(\mathrm{L})\int_a^x g_+^{'}(t)dt \qquad (13)$$

Since $(\mathrm{L})\int_a^x g_+^{'}(t)dt=0$, the identity (13) becomes, $\forall x \in [a,b),\ g(x)=g(a)$, implying that

$$g_+^{'}(x)=0\neq\chi(x).$$

This is a contradiction.

Corollary 2. Let $f:[a,b]\to R$ be a continuous function on the interval $[a,b]$. If the function $f_+^{'}$ is bounded on $[a,b)$, then the function f is almost everywhere differentiable on $[a,b)$, thus,

$$f^{'}(x)=f_+^{'}(x) \qquad (14)$$

(almost everywhere on $[a,b]$)

Proof. Since every absolutely continuous function is differentiable almost everywhere, the function f is almost differentiable on $[a,b)$.

Note. Moreover in [7] a stronger result than corollary 2 is mentioned. This is,

Suppose that a function f is right (left) differentiable at almost every point of $[a,b]$. Then f is differentiable almost everywhere.

We are going to prove a more general proposition, using the concept of right- hand derivative number.

Definition. *The number* $N_+f(x_0)$ *(finite or infinite) is called "a right*-hand derivative *number" of the function* f *at the point* x_0 *if there exists* a sequence of positive numbers $\{h_n\}$ such that $h_n\to 0$ and

$$\lim_{n\to\infty}\frac{f(x_0+h_n)-f(x_0)}{h_n}=N_+f(x_0) \qquad (15)$$

It is obvious that every function $f:[a,b]\to R$ at any point $x_0\in[a,b)$ has at least one right -hand derivative number. Indeed, we discern two cases. Firstly, if the sequence (16) is unlimited, then from this sequence we can retrieve a subsequence that converges towards ∞.

$$\frac{f(x_0+h_n)-f(x_0)}{h_n} \qquad (16)$$

Secondly, if the sequence (16) is limited, then from this sequence we are able to retrieve a convergent subsequence.

If we denote $N_+f(x)$ the set of all derivative numbers of the function f at x, then, in general, we can state that the function $x \mapsto N_+f(x)$ is not a single-value function.

Likewise, it is clear that the right- hand derivative at point x_0 of the function $f:[a,b]\to R$ exists at point x_0 if and only if all *right- hand derivative numbers of* the function f at x_0 are among themselves.

In addition, we need a generalization of the lemma 2, which can be found in [3]:

If the function $f:[a,b]\to R$ is continuous at $[a,b]$, then there exist at least two points c and d at (a,b) such that

$$N_+f(c)\leq\frac{f(b)-f(a)}{b-a}\leq N_+f(d) \qquad (17)$$

Where $N_+f(c)$ and $N_+f(d)$ are derivative numbers of the function f at the points c and d.

Theorem 3. Let $f:[a,b]\to R$ be a continuous function on the closed interval $[a,b]$, which has a right-hand derivative number $N_+f(t)$ at every point, $t\in[a,b)$. If the function N_+f is bounded on $[a,b)$, then the function $f^{'}$ is Lebesgue integrable on every interval $[a,x]\subset[a,b)$, according to,

$$(\mathrm{L})\int_a^x f^{'}(t)dt=f(x)-f(a) \qquad (18)$$

Proof. To prove this theorem, is sufficient to substitute $f_+^{'}(x)$ with $N_+f(x)$ in the proof of the theorem 2.

Corollary. If the function $N_+f(x)$ is bounded on $[a,b)$, then the function f is almost everywhere differentiable on $[a,b)$.

Note. Which conditions must be met by the right-hand derivative $f_+^{'}$ so that the formula (14) will hold true for every point $x\in[a,b)$?

To give an answer to the above question we use the following proposition (lemma 3), which can be found in [1] (see theorem 3, p. 242-243).

Lemma 3. If f is a measurable function, bounded and it has the Darboux property on $[a,b]$, then for each closed subinterval $\mathbf{I}=[p,q]\subset[a,b]$ there exists at least a point $\xi \in \mathbf{I}$ such tha

$$(\mathrm{L})\int_p^q f(x)dx = f(\xi)|\mathbf{I}| \qquad (|\mathbf{I}| = q-p) \tag{19}$$

The point ξ is called the mean point of $\mathbf{I}$ with respect to f.

Theorem 4. If the right-hand derivative $f_+^{'}$ of a continuous function $f:[a,b]\to R$ meets the following conditions:

1) $f_+^{'}$ is bounded, measurable and has the Darboux property on $[a,b)$, and

2) for each $x\in[a,b]$ and for each sequence of subintervals $\mathbf{I}_n=[p_n,q_n]\subset[a,b]$ that converges towards a point x $(\mathbf{I}_n \to x)$, we have $f_+^{'}(x_n)\to f_+^{'}(x)$, where x_n is the mean point of $\mathbf{I}_n$ with respect to f, then the function f is everywhere differentiable in $[a,b)$.

Proof. The function $f_+^{'}$ satisfies the conditions of theorem 4 according to,

$$(\mathrm{L})\int_a^x f_+^{'}(t)dt = f(x)-f(a) \tag{20}$$

Take $x\in[a,b)$, and let $\{h_n\}$ be an arbitrary nonzero sequence of real numbers such that $h_n\to 0$. Then

$$\lim_{n\to\infty}\frac{f(x+h_n)-f(x)}{h_n} = \lim_{n\to\infty}\frac{1}{h_n}\int_x^{x+h_n} f_+^{'}(t)dt =$$

$$= \lim_{n\to\infty}\frac{1}{h_n} f_+^{'}(x_n)(x+h_n-x) = \lim_{n\to\infty} f_+^{'}(x_n)$$

Thus, $f^{'}(x)=f_+^{'}(x)$, that means the theorem 4 holds.

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