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# THEOREM ON FIXED POINTS IN THREE COMPLETE METRIC SPACES

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**ABSTRACT**

A theorem on fixed points in three complete metric spaces is proved. We obtained our results based on achievments of Nešić [3] for two mappings of a space into itself. We have modified the methods used by Nešić [3] and by Jain, Shrivastava, Fisher [2]. The Theorem of Jain, Sahu, Fisher [1] is obtained as a corollary of our result..

**PЁRMBLEDHJE**

Në artikull jepet një teoremë për pikat fikse në tre hapësira të plota metrike. Në këtë rezultat arritëm duke u mbeshtetur ne rezultatin e Nešić [3] per dy pasqyrime të një hapësire në vetvete. Për vërtetimin e teoremës kemi modifikuar metodat e përdorura nga Nesiç në [3] dhe nga Jain, Shrivastava, Fisher në [2]. Rezultati që ne kemi fituar paraqet përgjithësim të teoremës Jain, Sahu, Fisher [1].

**Keywords:** fixed points, metric spaces, complete metric spaces.

## 1. INTRODUCTION

In 2003, the following theorem is proved by Nešić [3].

**Theorem 1.1 (**Nešić.**).** *Let S and T be mappings of the metric space (X,d) into itself satisfying the inequality:*

$$\left[1 + pd(x, y)\right]d(Sx, Ty) \leq$$
$$\leq p\left[d(x, Sx)d(y, Ty) + d(x, y)d(y, Sx)\right] +$$
$$+ q\max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\left[d(x, Ty) + d(y, Sx)\right]\right\}$$

*for all $x,y\in X$, where $p\geq 0$ and $0\leq q<1$.*
*If (X,d) is (T,S)-orbitally complete, then S and T have an unique fixed point u in X.*

Later, the following theorem is proved by Jain, Sahu, and Fisher [1].

**Theorem 1.2. (Jain, Sahu, Fisher [1])** *Let $(X,d_1)$, $(Y,d_2)$ and $(Z,d_3)$ be three complete metric spaces. If $T:X\to Y$, $S:Y\to Z$, $R:Z\to X$ from which at least two of them are continuous mappings, satisfying the following inequalities:*

$$d_1(RSTx, RSTx') \leq c\max\left\{d_1(x, x'), d_1(x, RSTx), d_1(x', RSTx'), d_2(Tx, Tx'), d_3(STx, STx')\right\}$$

$$d_2(TRSy, TRSy') \leq c\max\{d_2(y, y'), d_2(y, TRSy), d_2(y', TRSy'), d_3(Sy, Sy'), d_1(RSy, RSy')\}$$

$$d_3(STRz, STRz') \leq c\max\{d_3(z, z'), d_3(z, STRz), d_3(z', STRz'), d_1(Rz, Rz'), d_2(TRz, TRz')\}$$

*for all $x,x'\in X$, $y,y'\in Y$ and $z,z'\in Z$, where $0\leq c<1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z . Further, Tu=v, Sv=w and Rw=u.*

In this paper we will give a generalization of the Theorem of Jain, Sahu, Fisher [1].

## 2. MAIN RESULT

**Theorem 2.1.** *Let $(X,d_1)$, $(Y,d_2)$ and $(Z,d_3)$ be three complete metric spaces. If $T:X\to Y$, $S:Y\to Z$ and $R:Z\to X$ are mappings from which at least two of them are continuous and satisfying the following inequalities:*

$$\left[1+pd_1(x,RSTx')+pd_2(Tx,Tx')+pd_3(STx,STx')\right]d_1(RSTx,RSTx')\le$$
$$\le p\left[d_1(x,RSTx)d_3(STx,STx')+d_1(x,RSTx')d_3(STx',STx)+\right.$$
$$\left.+d_1(x,RSTx')d_2(Tx',Tx)+d_2(Tx',Tx)d_1(x,RSTx)\right] \quad (1)$$
$$+q\max\{d_1(x,x'),d_1(x,RSTx),d_1(x',RSTx'),$$
$$d_2(Tx,Tx'),d_3(STx,STx')\}$$

$$\left[1+pd_2(y,TRSy')+pd_3(Sy,Sy')+pd_1(RSy,RSy')\right]$$
$$d_2(TRSy,TRSy')\le p\left[d_2(y,TRSy)d_1(RSy,RSy')+\right.$$
$$+d_2(y,TRSy')d_1(RSy',RSy)+ \quad (2)$$
$$\left.+d_2(y,TRSy')d_3(Sy',Sy)+d_3(Sy',Sy)d_2(y,TRSy)\right]+$$
$$+q\max\{d_2(y,y'),d_2(y,TRSy),d_2(y',TRSy'),$$
$$d_3(Sy,Sy'),d_1(RSy,RSy')\}$$

$$\left[1+pd_3(z,STRz')+pd_1(Rz,Rz')+pd_2(TRz,TRz')\right]$$
$$d_3(STRz,STRz')\le p\left[d_3(z,STRz)d_2(TRz,TRz')+\right.$$
$$+d_3(z,STRz')d_2(TRz,TRz')+ \quad (3)$$
$$\left.+d_3(z,STRz')d_1(Rz',Rz)+d_1(Rz',Rz)d_3(z,STRz)\right]+$$
$$+q\max\{d_3(z,z'),d_3(z,STRz),d_3(z',STRz'),$$
$$d_1(Rz,Rz'),d_2(TRz,TRz')\}$$

*for all $x,x'\in X$, $y,y'\in Y$ and $z,z'\in Z$,, where $p\ge 0$, $0\le q<1$, then RST has an unique fixed point $\alpha\in X$, TRS has an unique fixed point $\beta\in Y$ and STR has an unique fixed point $\gamma\in Z$. Further:*

$$T\alpha=\beta,\ S\beta=\gamma,\ R\gamma=\alpha.$$

**Proof**. Let $x_0\in X$ be an arbitrary point. We define the three sequences $(x_n)$, $(y_n)$, $(z_n)$ with $X$, $Y$, $Z$ respectively as follows:

$$x_n=(RST)^n x_0,\ y_n=Tx_{n-1},\ z_n=Sy_n\ ;\ n=1,2,\ldots$$

Taking $y=y_n$ and $y'=y_{n-1}$ in (2), we obtain:

$$\left[1+pd_2(y_n,y_n)+pd_3(z_n,z_{n-1})+pd_1(x_n,x_{n-1})\right]$$
$$d_2(y_{n+1},y_n)\le p[d_2(y_n,y_{n+1})d_1(x_n,x_{n-1})+$$
$$+d_2(y_n,y_n)\times d_1(x_{n-1},x_n)+d_2(y_n,y_n)d_3(z_{n-1},z_n)+$$
$$+d_3(z_{n-1},z_n)d_2(y_n,y_{n+1})]$$
$$+q\max\{d_2(y_n,y_{n-1}),d_2(y_n,y_{n+1}),d_2(y_{n-1},y_n),$$
$$d_3(z_n,z_{n-1}),d_1(x_n,x_{n-1})\}$$

from which we take:

$$d_2(y_{n+1},y_n)\le q\max\{d_2(y_n,y_{n-1}),d_3(z_n,z_{n-1}),$$
$$d_1(x_n,x_{n-1}),d_2(y_n,y_{n+1})\}=$$
$$=q\max A$$

where $A=\{d_2(y_n,y_{n-1}),d_3(z_n,z_{n-1}),d_1(x_n,x_{n-1}),d_2(y_n,y_{n+1})\}$

If $\max A=d_2(y_{n+1},y_n)$, then

$$d_2(y_{n+1},y_n)\le qd(y_{n+1},y_n)$$

and since $0\le q<1$ it follows that $d_2(y_{n+1},y_n)=0$.

So, we always have:

$$d_2(y_{n+1},y_n)\le q\max\{d_1(x_n,x_{n-1}),d_2(y_n,y_{n-1}),d_3(z_n,z_{n-1})\} \quad (4)$$

In the same way, taking $z=z_n$ and $z'=z_{n-1}$ in (3) we obtain:

$$\left[1+pd_3(z_n,z_n)+pd_1(x_n,x_{n-1})+pd_2(y_{n+1},y_n)\right]$$
$$d_3(z_{n+1},z_n)\le p[d_3(z_n,z_{n+1})d_2(y_{n+1},y_n)+$$
$$+d_3(z_n,z_n)d_2(y_{n+1},y_n)+d_3(z_n,z_n)d_1(x_{n-1},x_n)+$$
$$+d_1(x_{n-1},x_n)d_3(z_n,z_{n+1})]+$$
$$+q\max\{d_3(z_n,z_{n-1}),d_3(z_n,z_{n+1}),d_3(z_{n-1},z_n),$$
$$d_1(x_n,x_{n-1}),d_2(y_{n+1},y_n)\}$$

So,

$$d_3(z_{n+1},z_n)\le q\max\{d_1(x_n,x_{n-1}),d_2(y_{n+1},y_n),d_3(z_n,z_{n-1})\}$$

and using (4), we get:

$$d_3(z_{n+1},z_n)\le q\max\{d_1(x_n,x_{n-1}),d_2(y_n,y_{n-1}),d_3(z_n,z_{n-1})\} \quad (5)$$

In the same way, taking $x=x_n$ and $x'=x_{n-1}$ in (1), we obtain:

$$[1+pd_1(x_n,x_n)+pd_2(y_{n+1},y_n)+pd_3(z_{n+1},z_n)]$$
$$d_1(x_{n+1},x_n)\le p[d_1(x_n,x_{n+1})d_3(z_{n+1},z_n)+$$
$$+d_1(x_n,x_n)d_3(z_n,z_{n+1})+d_1(x_n,x_n)d_2(y_n,y_{n+1})+$$
$$+d_2(y_n,y_{n+1})d_1(x_n,x_{n+1})+$$
$$+q\max\{d_1(x_n,x_{n-1}),d_1(x_n,x_{n+1}),d_1(x_{n-1},x_n),$$
$$d_2(y_{n+1},y_n),d_3(z_{n+1},z_n)\}$$

or

$$d_1(x_{n+1},x_n) \le q\max\{d_1(x_n,x_{n+1}),d_2(y_{n+1},y_n),$$
$$d_3(z_{n+1},z_n)\}$$

Using the inequalities (4) and (5), by the last inequality we obtain:

$$d_1(x_{n+1},x_n) \le q\max\{d_1(x_n,x_{n-1}),d_2(y_n,y_{n-1}),d_3(z_n,z_{n-1})\} \quad (6)$$

Taking $n=n-1$, $n-2$, …, using the inequalities (4), (5) and (6) we obtain:

$$d_1(x_n,x_{n+1}) \le q^{n-1}\max\{d_1(x_1,x_2),d_2(y_1,y_2),$$
$$d_3(z_1,z_2)\} = q^{n-1}\ell$$
$$d_2(y_n,y_{n+1}) \le q^{n-1}\max\{d_1(x_1,x_2),d_2(y_1,y_2),$$
$$d_3(z_1,z_2)\} = q^{n-1}\ell$$
$$d_3(z_n,z_{n+1}) \le q^{n-1}\max\{d_1(x_1,x_2),d_2(y_1,y_2),$$
$$d_3(z_1,z_2)\} = q^{n-1}\ell$$

where $\ell = \max\{d_1(x_1,x_2),d_2(y_1,y_2),d_3(z_1,z_2)\}$

So, the sequences $(x_n)$, $(y_n)$, $(z_n)$ are Cauchy sequences and they converge in $\alpha\in X$, $\beta\in Y$ and $\gamma\in Z$ respectively, since the metric spaces $(X,d_1)$, $(Y,d_2)$ and $(Z,d_3)$ are complete metric spaces.

Let suppose that $T$ and $S$ are continuous mappings. Then by

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_{n+1} \quad \text{and} \quad \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$$

we take:

$T\alpha=\beta$ and $S\beta=\gamma$. From them it follows that $ST\alpha=\gamma$.

Taking $x=\alpha$ and $x'=x_{n-1}$ using again the inequality (1) we obtain:

$$[1+pd_1(\alpha,x_n)+pd_2(T\alpha,y_n)+pd_3(ST\alpha,z_n)]$$
$$d_1(RST\alpha,x_n) \le p\big[d_1(\alpha,RST\alpha)d_3(ST\alpha,z_n)+$$
$$+d_1(\alpha,x_n)d_3(z_n,ST\alpha)+d_1(\alpha,x_n)d_2(y_n,T\alpha)+$$
$$+d_2(y_n,T\alpha)d_1(\alpha,RST\alpha)\big]+$$
$$+q\max\{d_1(\alpha,x_{n-1}),d_1(\alpha,RST\alpha),d_1(x_{n-1},x_n),$$
$$d_2(T\alpha,y_n),d_3(ST\alpha,z_n)\}$$

Letting $n$ tending in infinity and since $T\alpha=\beta$, $ST\alpha=\gamma$, we get:

$$d_1(RST\alpha,\alpha) \le q\max\{d_1(\alpha,RST\alpha)\}$$

from which it follows $d_1(RST\alpha)=\alpha$ , since $0\le q<1$. Thus, $\alpha$ is a fixed point of $RST$.

We also have:

$TRS\beta=TRST\alpha=T\alpha=\beta$ and $STR\gamma=STRS\beta=S\beta=\gamma$

Thus, $\beta$ is a fixed point of $TRS$ and $\gamma$ is a fixed point of $STR$.

Now let we show the unicity of $\alpha$. Let assume now that $RST$ has another fixed point $\alpha'$ different from $\alpha$.

Using the inequality (1), we get

$$d_1(\alpha,\alpha') = d_1(RST\alpha,RST\alpha') \le$$
$$\le p\frac{d_1(\alpha,\alpha)d_3(ST\alpha,ST\alpha')+d_1(\alpha,\alpha')d_3(ST\alpha',ST\alpha)}{1+pd_1(\alpha,\alpha')+pd_2(T\alpha,T\alpha')+pd_3(ST\alpha,ST\alpha')}+$$
$$+p\frac{d_1(\alpha,\alpha')d_2(T\alpha',T\alpha)+d_2(T\alpha',T\alpha)d_1(\alpha,\alpha)}{1+pd_1(\alpha,\alpha')+pd_2(T\alpha,T\alpha')+pd_3(ST\alpha,ST\alpha')}+$$
$$+q\frac{\max\{d_1(\alpha,\alpha'),d_1(\alpha,\alpha),d_1(\alpha',\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha,ST\alpha')\}}{1+pd_1(\alpha,\alpha')+pd_2(T\alpha,T\alpha')+pd_3(ST\alpha,ST\alpha')}$$

From which we get:

$$d_1(\alpha,\alpha') \le \frac{q}{1+pd_1(\alpha,\alpha')}\max\{d_1(\alpha,\alpha'),d_2(T\alpha,T\alpha'),$$
$$d_3(ST\alpha,ST\alpha')\}$$

and since $\frac{q}{1+pd_1(\alpha,\alpha')} \le q$ we get:

$$d_1(\alpha,\alpha') \le q\max\{d_1(\alpha,\alpha'),d_2(T\alpha,T\alpha'),d_3(ST\alpha,ST\alpha')\} = q\max A$$

If $\max A=d_1(\alpha,\alpha')$, then we get $d_1(\alpha,\alpha') \le qd_1(\alpha,\alpha')$, from which it follows $d_1(\alpha,\alpha')=0$.

Thus, $d_1(\alpha,\alpha') \le q\max\{d_2(T\alpha,T\alpha'),d_3(ST\alpha,ST\alpha')\}$ (7)

In the same way, using the inequality (2) we get:

$$d_2(T\alpha,T\alpha') = d_2(TRST\alpha,TRST\alpha') \le$$
$$\le q\max\{d_1(\alpha,\alpha'),d_3(ST\alpha,ST\alpha')\} \quad (8)$$

Using (7) and (8) we get

$$d_1(\alpha,\alpha') \le qd_3(ST\alpha,ST\alpha') \quad (9)$$

Using the inequality (3) for the right side of the inequality (9) we get:

$$d_1(\alpha,\alpha') \le qd_3(ST\alpha,ST\alpha') = qd_3(STRST\alpha,STRST\alpha') \le$$
$$\le q^2\max\{d_3(ST\alpha,ST\alpha'),d_1(\alpha,\alpha'),$$
$$d_2(T\alpha,T\alpha')\} = q^2 d_1(\alpha,\alpha')$$

Since $q<1$, we get $\alpha=\alpha'$. Thus, we proved the unicity of the fixed point $\alpha$ of $RST$.

In the same way it is proved the unicity of the fixed point $\beta$ of $TRS$ and the unicity of the fixed point $\gamma$ of $STR$.

Let we show that $R\gamma=\alpha$.

By the equalities

$$R\gamma = R(STR\gamma) = RST(R\gamma)$$

It follows that $R\gamma$ is a fixed point of $RST$. Since $\alpha$ is an unique common fixed point of $RST$, it follows that $R\gamma=\alpha$. This completes the proof of the theorem.

**Corollary 2.2** (Theorem Jain, Sahu, Fisher [2]).

*In case p=0, in Theorem 2.1, we obtain Theorem 1.2*

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