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GREEN'S RELATIONS IN PERIODIC Γ-SEMIGROUPS

KOSTAQ HILA AND JANI DINE
Department of Mathematics & Computer Sciences, Faculty of Natural Sciences, University of Gjirokastra, Albania
Email: kostaq_hila@yahoo.com

ABSTRACT

In this paper, we introduce and study periodic Γ-semigroups extending and generalizing the results obtained for periodic semigroups. We define a natural (equivalence) relation $\mathcal{K}$ in periodic Γ-semigroup that mimics the relation $\mathcal{K}$ in periodic semigroups studied by Schwarz. We investigate and characterize the periodic Γ-semigroups using Green's relations. The main purpose of this paper is to study the relationship between $\mathcal{K}$ and Green's relations, especially with the relation $\mathcal{D}$ and also with the relation $\mathcal{N}$.

Keywords: Γ-semigroup, Γ-group, Green's relation, simple, completely simple, weakly commutative, periodic

PËRMBLEDHJE

Në këtë punim, prezantojmë dhe studiojmë Γ-gjysmëgrupet periodikë duke zgjeruar dhe përgjithësuar rezultatet e përftuara për gjysmëgrupet periodikë. Përkufizojmë nje relacion (ekuivalence) natyral $\mathcal{K}$ në Γ-gjysmëgrupet perodikë i cili imiton relacionin $\mathcal{K}$ në gjysmëgrupet periodikë të studiuar nga Schwarz. Ne shqyrtojmë dhe karakterizojmë Γ-gjysmëgrupet periodikë duke përdorur relacionet e Green-it. Qëllimi kryesor i këtij punimi është të studiojë marrëdhënien midis relaconit $\mathcal{K}$ dhe relacioneve të Green-it, në veçanti me relacionin $\mathcal{D}$ dhe me relacionin $\mathcal{N}$.

Fjalë çelës: Γ-gjysmëgrup, Γ-grup, relacione të Green-it, i thjeshtë, plotësisht i thjeshtë, dobësisht ndërrues, periodic.

1. INTRODUCTION AND PRELIMINARIES

In 1981, Sen [12] introduced the concept and notion of the Γ-semigroup as a generalization of plain semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups in several papers. Green's relations for Γ-semigroups defined in [2, 14], play an important role in studying the structure of Γ-semigroups as well as in the case of the plain semigroups and have become a familiar tool among Γ-semigroups. Periodic semigroups have been treated occasionally in the literature with essential contribution provided by Schwarz [10], Yamada [18], Sedlock [11], Miller [8]. In this paper we will study periodic Γ-semigroups. We investigate and characterize the periodic Γ-semigroups using Green's relations. We define a natural (equivalence) relation $\mathcal{K}$ in periodic Γ-semigroup that mimics the relation $\mathcal{K}$ in periodic semigroups studied by Schwarz and study the relationship between $\mathcal{K}$ and Green's relations, especially with the relation $\mathcal{D}$ and also, with the relation $\mathcal{N}$. One can extend this work, searching other neccessary and sufficient conditions in order that the relation $\mathcal{K}$ coincides with any one of the other Green's relations.

In 1986, Sen and Saha [13] defined Γ-semigroup as a generalization of semigroup and ternary semigroup as follows:

Definition 1.1 *Let M and Γ be two nonempty sets. Then M is called a Γ-semigroup if there exists a mapping $M\times\Gamma\times M\rightarrow M$, written as $(a,\gamma,b)\mapsto a\gamma b$ satisfying the following identity $(a\alpha b)\beta c=a\alpha(b\beta c)$ for all a, b, $c\in M$ and for all α, $\beta\in\Gamma$.*

Example 1.2 *Let M be a semigroup and Γ be any nonempty set. Define a mapping $M\times\Gamma\times M\rightarrow M$ by $a\gamma b=b$ for all $a,b\in M$ and $\gamma\in\Gamma$. Then M is a Γ-semigroup.*

Example 1.3 *Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma=\{-\frac{1}{p}:p$ is prime$\}$. Let a, b, $c\in M$ and $\alpha\in\Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b then $a\alpha b\in M$ and $(a\alpha b)\beta c=a\alpha(b\beta c)$. Hence M is a Γ-semigroup.*

Example 1.4 *Let $M=\{-i,0,i\}$ and Γ=M. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.*

These examples show that every semigroup is a Γ-semigroup. Therefore, Γ-semigroups are a generalization of semigroups. Other examples of Γ-semigroups can be found in [4, 5, 13, 14, 17].

For nonempty subsets A and B of M and a nonempty subset Γ' of Γ, let $A\Gamma'B=\{a\gamma b: a\in A, b\in B$ and $\gamma\in\Gamma'\}$. If $A=\{a\}$, then we also write $a\Gamma'B$ instead of $\{a\}\Gamma'B$, and similarly if $B=\{b\}$ or $\Gamma'=\{\gamma\}$.

A Γ-semigroup M is called *commutative Γ-semigroup* if for all $a,b\in M$ and $\gamma\in\Gamma$, $a\gamma b=b\gamma a$. A nonempty subset K of a Γ-semigroup M is called a *sub-*Γ*-semigroup* of M if for all $a, b\in K$ and $\gamma\in\Gamma$, $a\gamma b\in K$.

Example 1.5 *Let* M= [0, 1] *and* $\Gamma=\{\frac{1}{n}|n$ *is a positive integer*$\}$. *Then M is a* Γ*-semigroup under usual multiplication. Let* K=[0,1/2]. *We have that K is a nonemtpy subset of M and* $a\gamma b\in K$ *for all* $a,b\in K$ *and* $\gamma\in\Gamma$. *Then K is a sub-Γ-semigroup of M .*

Definition 1.6 *Let M be a* Γ*-semigroup. A non-empty subset I of M is called a left (resp. right) ideal of M if* $M\Gamma I\subseteq I$ *(resp.* $I\Gamma M\subseteq I$*). A non-empty subset I of M is called an ideal of M if it is a left ideal as well as a right ideal of M.*

An element a of an Γ*-semigroup* M is called *idempotent* if $a=a\gamma a$, for some $\gamma\in\Gamma$. An element a of an Γ*-semigroup* M is called *zero element* of M if $a\gamma b=b\gamma a=a$, $\forall b\in M$ and $\forall\gamma\in\Gamma$ and it is denoted by 0. A Γ*-semigroup M* is called *left* (respectively, *right*) *simple* if it does not contain proper left (respectively, right) ideals or equivalently, if for every left (respectively, right) ideal A of M, we have $A=M$. The element a of a Γ*-semigroup M* is called *regular* in M if $a\in a\Gamma M\Gamma a$, where $a\Gamma M\Gamma a=\{(a\alpha b)\beta a\mid a,b\in M, \alpha,\beta\in\Gamma\}$. M is called regular if and only if every element of M is regular.

Definition 1.7 *A* Γ*-semigroup M with zero element is called 0-simple (left 0-simple, right 0-simple) if*

(1) $M\Gamma M\neq\{0\}$, *and*

(2) {0} *is the only proper two sided (left, right)-ideal of M.*

Definition 1.8 *A two sided (left, right) ideal I of a* Γ*-semigroup M is called* 0*-minimal if*

(1) $I\neq\{0\}$, *and*

(2) {0} *is the only two sided (left, right) ideal of M contained in I.*

For each element a of a Γ*-semigroup M*, the left ideal $M\Gamma a\cup\{a\}$ containing a is the smallest left ideal of M containing a, for if A is any other left ideal containing a, then $M\Gamma a\cup\{a\}\subseteq A$ and this ideal is denoted by $(a)_l$ and called the principal left ideal generated by the element a. Similarly for each $a\in M$, the smallest right ideal containing a is $a\Gamma M\cup\{a\}$ which is denoted by $(a)_r$ and called the principal right ideal generated by the element a. The principal ideal of M generated by the element a is denoted by (a) and $(a)=\{a\}\cup M\Gamma a\cup\{a\}\Gamma M\cup M\Gamma\{a\}\Gamma M$.

Let M be a Γ-semigroup and x be a fixed element of Γ. We define $a\circ b$ in M by $a\circ b=axb, \forall a,b\in M$. The authors [13] have shown that M is a semigroup and denoted this semigroup by M_x. They have shown that if M_x is a group for some $x\in\Gamma$, then M_x is a group for all $x\in\Gamma$. A Γ-semigroup M is called a Γ*-group* if M_x is a group for some (hence for all) $x\in\Gamma$.

In [10] the authors defined the Green's equivalences on Γ-semigroup as follows:

Let M be a Γ-semigroup. Let $a,b\in M$,
$a\mathcal{L}b\Leftrightarrow(a)_l=(b)_l$, $a\mathcal{R}b\Leftrightarrow(a)_r=(b)_r$, $a\mathcal{J}b\Leftrightarrow(a)=(b)$,
$a\mathcal{H}b\Leftrightarrow a\mathcal{L}b$ and $a\mathcal{R}b$, $a\mathcal{D}b\Leftrightarrow a\mathcal{L}c$ and $c\mathcal{R}b$ for some $c\in M$.

It is clear that a Γ-semigroup M is left [right] simple if and only if it consists of a single $\mathcal{L}[\mathcal{R}]$ class, and that M is simple if and only if it consists of a single $\mathcal{J}$-class. We say that a Γ-semigroup M is *bisimple* if it consists of a single $\mathcal{D}$-class. Since $\mathcal{D}\subseteq\mathcal{J}$, every bisimple Γ-semigroup is also simple.

Definition 1.9 *Let M a* Γ-semigroup. *If* L_a *and* L_b *are* $\mathcal{L}$*-classes containing a and b of a* Γ-semigroup M *respectively, then* $L_a\leq L_b$ *if* $(a)_l\subseteq(b)_l$. *Then "*$\leq$*"is a partial order in* $M/\mathcal{L}$ *which is the set of* $\mathcal{L}$*-classes of M. Similarly* $R_a\leq R_b$ *and* $J_a\leq J_b$ *are defined in* $M/\mathcal{R}$ *and* $M/\mathcal{J}$.

Definition 1.10 *A* Γ-semigroup *M is said to satisfy* min_L *or* min_R *condition if every nonempty set of* $\mathcal{L}$*-classes or of* $\mathcal{R}$*-classes possess a minimal member respectively.*

Definition 1.11 *A* Γ-semigroup *M is called completely 0-simple if M is 0-simple and it satisfies the* min_L *and* min_R *conditions.*

2. The natural equivalence and $\mathcal{D}$ A Γ-semigroup M is said to be a *periodic Γ-semigroup* [10] if for any $a\in M$ and any $\gamma\in\Gamma$, there exist positive integers n and m such that $(a\gamma)^n b=(a\gamma)^{n+m}b$ and $b(\gamma a)^n=b(\gamma a)^{n+m}$ for all $b\in M$. Equivalently, a Γ-semigroup M is said to be a *periodic Γ-semigroup* if each element of M has a finite order, where the order of $a\in M$ is the order of the cyclic sub-Γ-semigroup of M generated by a, that is, to each element a of M, for all $\gamma\in\Gamma$, there corresponds an idempotent e and a positive integer n such that $(a\gamma)^{n-1}a=e$ for all $\gamma\in\Gamma$; the element a is then said to *belong to e*.

The fact that for each element a of a periodic Γ-semigroup M some power of a is idempotent leads to defining a natural (equivalence) relation $\mathcal{K}$ on M by: for $a,b\in M$, $a\mathcal{K}b$ if and only if for all $\gamma\in\Gamma$, there exists an idempotent e and integers m, n such that $(a\gamma)^{n-1}a=(b\gamma)^{m-1}b=e$. The $\mathcal{K}$-classes of M will be denoted by K^e, e idempotent.

Let e be an idempotent. A sub-Γ-group G of Γ-semigroup M is called *maximal sub-Γ-group* to belong to e if a) $e\in G$; and b) it is not properly contained in any other sub-Γ-group of M. This will be denoted by G^e. For the set G^e we have obviously the following facts:

I) for each α-idempotent e, $e\gamma x = x\gamma e = x$ for each $x\in K^e$, $\gamma\in\Gamma$, and $e\alpha K^e=K^e\alpha e=G^e=\{x\in K^e \mid e\alpha x=x\alpha e=x\}$ is the maximal sub-Γ-group of M containing e;

II) M is a union of sub-Γ-groups if and only if each element of M has index one if and only if for each idempotent e, $K^e=G^e$.

III) for each idempotent e, $H_e=G^e$.

In [2], the authors proved the following:

Theorem 2.1 [2, Theorem 3.4] *If M is a periodic* Γ-semigroup*, then* $\mathcal{D}=\mathcal{J}$.

We will determine necessary and sufficient conditions on a periodic Γ-semigroup M in order that $\mathcal{K}$ coincides with any one of the Green relations. It is easily verifiable the following theorem.

Theorem 2.2 *For each idempotent e in a periodic* Γ-semigroup *M, $K^e\cap D_e=G^e$.*

Corollary 2.3 *For each idempotent e in a periodic* Γ-semigroup*, $K^e\cap L_e=G^e$ and $K^e\cap R_e=G^e$.*

Corollary 2.4 *Periodic* Γ-semigroup *M is a union of Γ-subgroups if and only if $\mathcal{K}\subseteq\mathcal{D}$.*

Definition 2.5 Γ-semigroup *M is weakly comutative if for each $a,b\in M$ and $\alpha,\gamma\in\Gamma$, there exist $x,y\in M$ and an integer k such that $((a\alpha b)\gamma)^{k-1}(a\alpha b)=x\alpha a=b\alpha y$.*

Definition 2.6 Γ-semigroup *M is a semilattice of* Γ-*semigroups of type α if M is a disjoint union of* Γ-*semigroups of type α$\{M_i \mid i\in I$, I indexset$\}$, and for each i, $j\in I$ there exists $k\in I$ such that $M_i\Gamma M_j\subseteq M_k$ and $M_j\Gamma M_i\subseteq M_k$.*

Let M be a Γ-semigroup and F a sub-Γ-semigroup. Then F is called a *filter* of M if $a,b\in M$, $a\gamma b\in F(\gamma\in\Gamma)\Rightarrow a\in F$ and $b\in F$ [4,15]. It is clear that for every $a\in M$ there is a unique smallest filter of M containing the element a, denoted by $N(a)$, which is called the principal filter generated by a. We denote by "$\mathcal{N}$" the equivalence relation on M defined by $\mathcal{N}=\{(a,b)\in M^2 \mid N(a)=N(b)\}$. $\mathcal{N}$ is a semilattice congruence on M. For any $a\in M$, the $\mathcal{N}$-class containing a is denoted by $(a)_{\mathcal{N}}$ and it is clear that it is a sub-Γ-semigroup of M. On the set $M/\mathcal{N}=\{(a)_{\mathcal{N}} \mid a\in M\}$ we define $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}}=(a\gamma b)_{\mathcal{N}}$, for all $(a)_{\mathcal{N}},(b)_{\mathcal{N}}\in M/\mathcal{N},\gamma\in\Gamma$. It is clear that the set $M/\mathcal{N}$ is a Γ-semigroup.

Theorem 2.7 *Let M be a* Γ-semigroup*. For every $x\in M$, $N(x)=\{y\in M \mid \langle x\rangle\cap M\Gamma y\cap y\Gamma M\neq\emptyset\}$ if and only if M is weakly commutative.*

Proof. We first prove the necessity. Let $x,y\in M$. Then, for $\alpha\in\Gamma$, $x\in N(x)\subseteq N(x\alpha y)=\{z\in M \mid \langle x\alpha y\rangle\cap M\Gamma z\cap z\Gamma M\neq\emptyset\}$ and thus $((x\alpha y)\gamma)^{m-1}(x\alpha y)=a\alpha x$ for some $a\in M$ and some integer m; similarly, $((x\alpha y)\gamma)^{n-1}(x\alpha y)=y\alpha b$ for some $b\in M$ and some integer n. If $m>n$, then $((x\alpha y)\gamma)^{m-1}(x\alpha y)=a\alpha x=y\alpha[b\gamma((x\alpha y)\gamma)^{m-n}(x\alpha y)]$; the other cases are similar. Hence M is weakly commutative.

We next prove the sufficiency. Let $x\in M$ and let $T=\{y\in M \mid \langle x\rangle\cap M\Gamma y\neq\emptyset\}$.

We first show that T is filter of M; this together with the fact that $x\in T$ will prove that $N(x)\subseteq T$. If $y,z\in T$, then for all $\gamma\in\Gamma$, $(x\gamma)^{m-1}x=a\alpha y$ and $(x\gamma)^{n-1}x=b\beta z$ for some $a,b\in M$, $\alpha,\beta\in\Gamma$ and for some integer m, n. Since M is weakly commutative, $(x\gamma)^{nr-1}x=((b\beta z)\gamma)^{r-1}(b\beta z)=z\delta c$ for some $c\in M$, $\delta\in\Gamma$ and for some integer r. Consequently $(x\gamma)^{(m+nr)-1}x=[a\alpha(y\gamma z)]\delta c$, whence, again by weak commutativity, $(x\gamma)^{(m+nr)k-1}x=d\rho[a\alpha(y\gamma z)]$ for some $d\in M$, $\rho\in\Gamma$ and for some integer k. Thus, $y\gamma z\in T$ for all $\gamma\in\Gamma$. Conversely, suppose that $y\gamma z\in T$ for all $\gamma\in\Gamma$. Hence $(x\gamma)^{m-1}x=(a\alpha y)\gamma z$ for some $a\in M$, $\alpha\in\Gamma$ and for some integer m, and thus $z\in T$. It follows by weak commutativity that $(x\gamma)^{mn-1}x=([(a\alpha y)\gamma z]\gamma)^{n-1}=([(a\alpha y)\gamma z]= =b\beta(a\alpha y)$ for some $b\in M$, $\beta\in\Gamma$ and for some integer n; that is, $y\in T$. Hence T is a filter and $N(x)\subseteq T$. Since the opposite inclusion is clearly satisfied, we have $N(x)=T$.

By symmetry we conclude that also $N(x)=\{y\in M \mid \langle x\rangle\cap y\Gamma M\neq\emptyset\}$. It is easily to see that the set $\{y\in M \mid \langle x\rangle\cap y\Gamma M\neq\emptyset\}\cap\{y\in M \mid \langle x\rangle\cap M\Gamma y\neq\emptyset\}$ is the one in the statement of the theorem.

Theorem 2.8 *Let M be a periodic* Γ-semigroup*. Then every K^e is an $\mathcal{N}$-class of M if and only if M is weakly commutative.*

Proof. Suppose that every K^e is an $\mathcal{N}$-class of M. If x, $y\in M$, then $x\alpha y, y\alpha x\in(x\alpha y)_{\mathcal{N}}=K^f$ for all $\alpha\in\Gamma$ and for some γ-idempotent f. Thus, $((x\alpha y)\gamma)^{m-1}(x\alpha y)= =((y\alpha x)\gamma)^{n-1}(y\alpha x)=f$ for some m and n. Hence $((x\alpha y)\gamma)^{m-1}(x\alpha y)=((y\alpha x)\gamma)^{2n-1}(y\alpha x)=y\alpha[x\gamma((y\alpha x)\gamma)^{2n-1}]= =[((y\alpha x)\gamma)^{2n-1}(y\alpha x)\gamma y]\alpha x$.

Conversely, suppose M is weakly commutative. For any $x\in M$, $(x)_{\mathcal{N}}$ is Γ-semigroup and thus contains an idempotent. If $e,f\in E_\gamma\cap(x)_{\mathcal{N}}$, then $e=a\gamma f$ and $f=e\gamma b$ for some a, $b\in M$ by Theorem 2.7. Consequently, $e=a\gamma f=(a\gamma f)\gamma f=e\gamma f=e\gamma(e\gamma b)=e\gamma b=f$. Hence $(x)_{\mathcal{N}}\subseteq K^e$ and since the opposite inclusion is obvious, we conclude that $(x)_{\mathcal{N}} = K^e$.

Proposition 2.9 *If M is a weakly commutative periodic* Γ*-semigroup, then* $\mathcal{D} \subseteq \mathcal{K} = \mathcal{N}$ *and each maximal sub-Γ-group is a* $\mathcal{D}$ *-class of M.*

Proof. It can be easily verified. We omit it.

Definition 2.10 *A* Γ*-semigroup M is called unipotent if it contains exactly one idempotent.*

Definition 2.11 *An element u of a* Γ*-semigroup M is called a zeroid element of M if, for each element a of M, there exist* $x,y \in M$, $\alpha, \beta \in \Gamma$ *such that* $a\alpha x = y\beta a = u$.

Definition 2.12 *Γ-homogroup is called a Γ-semigroup having zeroid elements.*

Let M be a periodic Γ-semigroup. We introduce the following three conditions which will play principal roles in sequel.

Condition A. For any elements $a, b \in M$ and $\gamma \in \Gamma$, if $(a\gamma)^{n-1}a=(b\gamma)^{m-1}b$ for some integer *n, m*, then for all $\alpha \in \Gamma$, there exist three positive integer *r, s* and *t* such that $((a\alpha b)\gamma)^{r-1}(a\alpha b)=(a\gamma)^s a\alpha(b\gamma)^t b=(b\gamma)^t b\alpha(a\gamma)^s a$.

Condition B. For any elements $a,b \in M$, and for any positive integers *n, m* and $\alpha, \gamma \in \Gamma$, then there exist two positive integer *r, s* such that $((a\alpha b)\gamma)^r(a\alpha b)= =((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b)\gamma)^{s-1}((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b))$.

Condition C. For any elements $a,b \in M$, and for any positive integers *n, m* and $\alpha, \gamma \in \Gamma$, then there exist two positive integer *r, s* such that $((a\alpha b)\gamma)^{r-1}(a\alpha b)= =(((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b)\gamma)^{s-1}((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b))= =(((b\gamma)^{m1}b\alpha(a\gamma)^{n-1}a)\gamma)^{t-1}(b\gamma)^{m-1}b\alpha(a\gamma)^{n-1}a)$.

Lemma 2.13 *For periodic Γ-semigroups, Condition A is a consequence of Condition B.*

Proof. It can be easily verified. We omit it.

Lemma 2.14 *For periodic Γ-semigroups, Condition B is a consequence of Condition C.*

Proof. It is immediate from the definitions of Conditions B and C.

Theorem 2.15 *A periodic Γ-semigroup M is decomposable into the class sum of mutually disjoint unipotent Γ-homogroups if and only if it satisfies Condition A. Further, in this case such a decomposition is uniquely determined.*

Proof. Let *M* satisfies Condition A. To prove the "*if*" part of this theorem, we need to show only that each K^e is a Γ-homogroup. If *a, b* are two elements of K^e, then there exist integers *n, m* such that $(a\gamma)^{n-1}a=(b\gamma)^{m-1}b=e$. Since *M* satisfies Condition A, there exist three positive integer *r, s* and *t* such that $((a\alpha b)\gamma)^{r-1}(a\alpha b)=(a\gamma)^{s-1}a\alpha(b\gamma)^{t-1}b=(b\gamma)^{t-1}b\alpha(a\gamma)^{s-1}a$.
Hence, $((a\alpha b)\gamma)^{rnm-1}(a\alpha b)=(a\gamma)^{snm-1}a\alpha(b\gamma)^{tnm-1}b=e$. This implies that K^e is Γ-semigroup. Since *e* is clearly a zeroid element of K^e, the semigroup K^e is a Γ-homogroup.

Conversely, assume that M is decomposed into the class sum of mutually disjoint unipotent Γ-homogroups H_i and suppose that $(a\gamma)^{n-1}a=(b\gamma)^{m-1}b$. Then both *a* and *b* are contained in the same Γ-homogroup, say H_i, since there exists an idempotent *e* and an integer *s* such that $(a\gamma)^{ns-1}a=(b\gamma)^{ms-1}b=e$. Therefore, $a\alpha b \in H_i$, and,

$$(a\gamma)^{ns-1}a\alpha(b\gamma)^{ms-1}b=(b\gamma)^{ms-1}b\alpha(a\gamma)^{ns-1}a=e= =(((a\alpha b)\gamma)^{r-1}(a\alpha b).$$

for some integer *r*. Thus, the proof of the first half of this theorem is complete. The latter half of this theorem is clear.

A *band* is an idempotent Γ-semigroup. Let *J* be a band. A Γ-semigroup G is said to be a band *J* of Γ-semigroups of type *T*, if G is the class sum of a set $\{G_i \mid i \in J\}$ of mutually disjoint sub-Γ-semigroup G_i each type T, such that for any $i,j \in J$, $G_i\Gamma G_j \subset G_{i\gamma j}$, $\gamma \in \Gamma$. If *J* is a commutative band, that is, if *J* is a semilattice, then M is called a semilattice of Γ-semigroups of type T.

Theorem 2.16 *A periodic Γ-semigroup M is decomposable into a band of unipotent Γ-homogroups if and only if it satisfies Condition B. Further, in this case such a decomposition is uniquely determined*.

Proof. Assume that M is a band of unipotent Γ-homogroup H_i. Let $a, b \in M$ *and n, m* be any integers. Then, there exist H_i and H_j which contain *a* and *b* respectively. Since, by the assumption on *M*, both $a\alpha b$ and $(a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b$ are contained in $H_{i\alpha j}$, there exist integers *r* and *s* such that $((a\alpha b)\gamma)^{r-1}(a\alpha b)=e_{i\alpha j}$ where $e_{i\alpha j}$ is the idempotent of $H_{i\alpha j}$. *Conversely*, let M satisfies Condition B. By Theorem 2.15, M is the class sum of mutually disjoint unipotent Γ-homogroups H_i, since M satisfies also Condition A. Pick up any $a_1, a_2 \in H_i$ and $b_1, b_2 \in H_j$ respectively. There exist integers n_1, n_2, m_1, m_2 such

$$(a_1\gamma)^{n_1-1}a_1 = e_i, (a_2\gamma)^{n_2-1}a_2 = e_i, (b_1\gamma)^{m_1-1}b_1 = e_j, (b_2\gamma)^{m_2-1}b_2 = e_j$$

where e_i, e_j are idempotents of H_i and H_j respectively. By Condition B, we have

$$((a_1\alpha b_1)\gamma)^{r_1-1}(a_1\alpha b_1) = ((e_i\alpha e_j)\gamma)^{s_1-1}(e_i\alpha e_j) \quad \text{and}$$

$$((a_2\alpha b_2)\gamma)^{r_2-1}(a_2\alpha b_2) = ((e_i\alpha e_j)\gamma)^{s_2-1}(e_i\alpha e_j)$$

for some integers r_1, s_1, r_2, s_2.

If $e_i\alpha e_j \in H_t$, there exists an integer *n* such that $((e_i\alpha e_j)\gamma)^{n-1}(e_i\alpha e_j)=e_t$, where e_t is the idempotent of H_t. Therefore, we have

$$((a_1\alpha b_1)\gamma)^{r_1 n-1}(a_1\alpha b_1) = ((e_i\alpha e_j)\gamma)^{s_1 n-1}(e_i\alpha e_j) = e_t =$$
$$= ((e_i\alpha e_j)\gamma)^{s_2 n-1}(e_i\alpha e_j) = ((a_2\alpha b_2)\gamma)^{r_2 n-1}(a_2\alpha b_2).$$

This implies that both $a_1\alpha b_1$ and $a_2\alpha b_2$ are contained in H_t, that is, $H_i\Gamma H_j \subset H_t$. Thus, the proof of the first half of this theorem is complete. The latter half of the theroem follows from Theorem 2.15.

Theorem 2.17 *A periodic Γ-semigroup M is decomposable into a semilattice of unipotent*

Γ-homogroups if and only if it satisfies Condition C. Further, in this case such a decomposition is uniquely determined.

Proof. Let M satisfies Condition C. By Lemma 2.14, M satisfies Condition B. Therefore, by Theorem 2.16, M is decomposed uniquely into a band J of unipotent Γ-homogroups H_i. Let $a, b \in M$. Then, by Condition C, there exist two integers s, t such that $((a\alpha b)\gamma)^{s-1}(a\alpha b)=((b\alpha a)\gamma)^{t-1}(b\alpha a)$. Hence, there exists H_i which contains both $a\alpha b$ and $b\alpha a$. This implies that J is a semilattice.

Conversely, assume that M is decomposable into a semilattice of unipotent Γ-homogroups H_i. Let $a, b \in M$ and n, m be any integers. Since M satisfies Condition B, there exist integers r_1, r_2, r_3, r_4 such that

$$((a\alpha b)\gamma)^{r_1-1}(a\alpha b)=((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b)\gamma)^{r_2-1}((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b))$$

and

$$((b\alpha a)\gamma)^{r_3-1}(b\alpha a)=((b\gamma)^{m-1}b\alpha(a\gamma)^{n-1}a)\gamma)^{r_4-1}((b\gamma)^{m-1}b\alpha(a\gamma)^{n-1}a)).$$

On the other hand, it follows from our assumption on M that $a\alpha b$ and $b\alpha a$ are contained in the same unipotent Γ-homogroup, say H_i under the decomposition. Hence, there exist two integers t_1, t_2 such that $((a\alpha b)\gamma)^{t_1-1}(a\alpha b)=((b\alpha a)\gamma)^{t_2-1}(b\alpha a)=e_i$, where e_i is the idempotent of H_i. Consequently, we have

$$((a\alpha b)\gamma)^{t_1t_2r_1r_3-1}(a\alpha b)=$$
$$=((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b)\gamma)^{t_1t_2r_2r_3-1}((a\gamma)^{n-1}a\alpha(b\gamma)^{m-1}b))=$$
$$=((b\gamma)^{m-1}b\alpha(a\gamma)^{n-1}a)\gamma)^{t_1t_2r_1r_4-1}((b\gamma)^{m-1}b\alpha(a\gamma)^{n-1}a)).$$

Lemma 2.18 *Periodic Γ-semigroup M is weakly commutative if and only if it satifies Condition C.*

Proof. It can be easily verified. We omit it.

To prove the main theorem, we need some other results as follows. As an application of the results proved in Theorem 1, Theorem 2 and Theorem 3 in [7] and Proposition 2.21 in [6], we have the following:

Lemma 2.19 *A Γ-semigroup M is left [right, intra-] regular if and only if every left [right, two-sided] ideal of M is semiprime.*

Theorem 2.20 *The following are equivalent conditions on a Γ-semigroup M.*

(1) *M is left regular.*

(2) *Every left ideal of M is semiprime*.

(3) *Every $\mathcal{L}$-class of M is a left simple sub-Γ-semigroup of M.*

(4) *Every left $\mathcal{L}$-class of M is a sub-Γ-semigroup of M.*

(5) *M is a disjoint union of left simple sub-Γ-semigroups.*

(6) M *is a union of left simple sub-Γ-semigroups.*

Theorem 2.21 *The following are equivalent conditions on a Γ-semigroup M.*

(1) M *is a union of Γ-groups.*

(2) M *is both left and right regular.*

(3) *Every left and every right ideal of M is semiprime.*

(4) *Every $\mathcal{H}$-class of M is a Γ-group.*

(5) *M is a union of disjoint Γ-groups.*

Proof. If (1) holds, then M is clearly left regular, right regular and regular; for we may solve $x\gamma a\mu a=a$, $a\mu a\gamma y=a$, $a\gamma z\mu a=a$ for some $\gamma, \mu \in \Gamma$ and x,y,z within a sub-Γ-group of M to which a belongs. Thus (1) implies (2). Moreover, (2) is equivalent to (3) by Lemma 2.19.

By definition of left and right regular Γ-semigroup and by Greens' Theorem for Γ-semigroups [9, Theorem 2.1], it follows that (4) holds. (4) implies (5) since $\mathcal{H}$-classes are disjoint, and (5) implies (1) trivially. So, we have established the equivalence of (1), (2), (3), (4) and (5).

Theorem 2.22 *The following four statements are equivalent.*

(1)M *is a union of simple Γ-semigroups.*

(2) M *is intra-regular.*

(3) *Every ideal of M is semiprime.*

(4) *The principal ideals of M constitute a semilattice Y under intersection; in fact $J(a)\cap J(b)=J(a\gamma b)$ for every $a, b\in M$, $\gamma\in\Gamma$; furthermore, M is the union of the semilattice Y of simple Γ-semigroups $M_i(i\in Y)$, each M_i being a $\mathcal{J}$-class of* M.

Proof. By Lemma 2.19, Theorem 2.20 and Theorem 2.21 it follows that (1) implies (2) and (2) is equivalent to (3). Evidently (4) implies (1). The proof of the fact that (4) follows from (2) and (3) can be done by easily respectively modifications in the last part of the proof of Theorem 4.4 in [**Error! Reference source not found.**].

Theorem 2.23 *The following statements are mutually equivalent.*

(1) *M is union of Γ-groups.*

(2) M *is a union of completely simple Γ-semigroups.*

(3) M *is a semilattice Y of completely simple Γ-semigroups $M_i(i\in Y)$, where Y is the semilattice of principal ideals of M, and each M_i is a $\mathcal{J}$-class of M.*

Proof. (3) implies (2) trivially and (2) implies (1) by Lemma 2.9 [16] and Theorem 2.1 [9]. By Theorem 2.22, (1) implies (3) except for the complete simplicity of the simple sub-Γ-semigroup M_i. This is immediate from the Theorem 4.4 [16], since each $\mathcal{J}$-class M_i is a union of the $\mathcal{H}$-classes of M contained in it, while (1) implies that every $\mathcal{H}$-classes of M is a Γ-group from Theorem 2.21.

Now we prove the main theorem of this section.

Theorem 2.24 *Let M be a periodic Γ-semigroup. Then the following are equivalent:*

(1) M *is a semilattice of Γ-subgroups.*

(2) M *is a union of Γ-subgroups and weakly commutative.*

(3) $\mathcal{K}=\mathcal{D}$.

Proof. $(1)\Rightarrow(2)$. Since M is a semilattice of (periodic) Γ-subgroups, then it is trivially both a union of Γ-subgroups and a semilattice of unipotent Γ-homogroups. Due to Theorem 2.17, M satisfies condition Ψ, and by Lemma 2.18 it is weakly commutative.

$(2)\Rightarrow(3)$. If M is a union of Γ-subgroups, then for each idempotent *e*, $K^e=G^e=H_e$. Thus $\mathcal{K}=\mathcal{H}$. Therefore, combining this with all the above mentioned results, we get $\mathcal{D}=\mathcal{J}\subseteq\mathcal{N}=\mathcal{K}=\mathcal{H}$. However, it follows from the definitions of Green's relations that $\mathcal{H}\subseteq\mathcal{L}\subseteq\mathcal{D}$ and $\mathcal{H}\subseteq\mathcal{R}\subseteq\mathcal{D}$. So all of Green's relations coincide with $\mathcal{K}$.

$(3)\Rightarrow(1)$. From Corollary 2.4, $\mathcal{K}\subseteq\mathcal{D}$ implies that M is union Γ-subgroups, that is, each $\mathcal{K}$-class is a maximal Γ-group. By Theorem 2.23, M is a semilattice of $\mathcal{D}=\mathcal{H}$-classes.

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