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ACCELERATING SIMULTANEOUS METHODS FOR THE DETERMINATION OF POLYNOMIAL MULTIPLE ROOTS

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SUMMARY
The aim of this paper is to develop improved methods for finding the multiple and simple roots of polynomial equations. The improved methods are based on Durand-Kerner method (a modification of Weierstrass method) and Laguerre method. The new simultaneous method has higher order of convergence and high computational efficiency since the accelerated convergence is attained with only negligible number of additional numerical operations. We have studied two classes of algorithms for solving polynomial equations: those with a known order of multiplicity and others with no information on multiplicity. For the second class we have proposed a suitable algorithm that computes simultaneously the distinct roots of polynomials and their respective multiplicities.
Key words: order of convergence, multiple roots, simple roots, polynomial, GCD.

1. INTRODUCTION

Laguerre's method belongs to the most powerful methods for solving polynomial equations. Two modifications of Laguerre's method, which enable simultaneous determination of all simple zeros of a polynomial and posses the convergence rate at least four, were proposed by Hansen, Patrick and Rusnak [1]. In Section 2 we present a fixed point relation of Laguerre's type, which is concerned with multiple zeros of a polynomial. A significant improvement of computational efficiency of simultaneous methods can be achieved by using suitable correction terms. Such an approach, based on Nourein's idea [11] for the simultaneous methods, was applied for the first time in [12, M. S. Petrović, C. Carstensen, On some improved inclision methods for polynomial roots with Weierstrass' correction, Comput. Math. Appl. 25 (1993), 73-82] to the Börsch-Supan-like method. Based in this idea, in Section 3 we develop a new simultaneous method for finding simple roots of polynomials, modifing the Weierstrass function. The developing of this last method for finding simple roots of polynomials is related to the fact that in Section 3 we also discuss the problem when the multiplicity of the root is not known, and the technique we use is deflating the multiple roots into simple ones and implement the proposed simultaneous method for computing them.

2. SIMULTANEOUS METHODS FOR FINDING MULTIPLE ZEROS

Let P be a monic polynomial of degree n with multiple zeros $\lambda_1,\cdots,\lambda_k$ $(k \le n)$ of the respective multiplicities $m_1,\cdots,m_k$

$$P(z)=\prod_{j=1}^{k}(z-\lambda_j)^{m_j}. \quad (2.1)$$

We shall not consider in this section the problem of determining the order of multiplicity, to leave space for it in the next section.

For the point $z=z_i$ $\left(i \in I_k=\{1,\cdots,k\}\right)$ let us introduce the notations:

$$\Sigma_{l,i} = \sum_{\substack{j=1\\ j\neq}}^{k} \frac{m_j}{(z_i \quad \lambda_j)^l} \quad (l=1,2), \qquad {}_i = n\,\Sigma_{2,i} \quad \frac{n}{n \quad m_i}\Sigma_{1,i}^2 \ ,$$

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \ \delta_{2,i} = \frac{P'(z_i)^2 \quad P(z_i)P''(z_i)}{P(z_i)^2},$$

$$\varepsilon_i = z_i \quad \lambda_i \, .$$

In [5, M. Petrović, L. Rancić, D. Milosević, Laguerre-like methods for the simultaneous approximation of polynomial multiple zeros, Yugoslav Journal of Operational Research, 16 (2006), Number 1, 31-44] is derived the following fixed point relation

$$\lambda_i = z_i \quad \frac{n}{\delta_{1,i} \pm \sqrt{\frac{n \quad m_i}{m_i}\left(n\delta_{2,i} \quad \delta_{1,i}^2 \quad {}_i\right)}}$$

$$= z_i \quad \frac{n}{\delta_{1,i} \pm \sqrt{\frac{n \quad m_i}{m_i} \quad n\delta_{2,i} \quad \delta_{1,i}^2 \quad n\,\Sigma_{2,i} + \frac{n}{n \quad m_i}\Sigma_{1,i}^2}},$$

(2.2)

which is suitable for the construction of iterative methods for the simultaneous finding multiple zeros of a given polynomial in ordinary complex arithmetic as well as complex interval arithmetic. If we substitute the exact zeros appearing in the sums $\Sigma_{1,i}$ and $\Sigma_{2,i}$ by their approximations, we obtain the sums

$$S_{1,i} = \sum_{\substack{j=1\\ j\neq}}^{k} \frac{m_j}{z_i \quad z_j}, \quad S_{2,i} = \sum_{\substack{j=1\\ j\neq}}^{k} \frac{m_j}{(z_i \quad z_j)^2}$$

which are some approximations to $\Sigma_{1,i}$ and $\Sigma_{2,i}$. Then

$$f_i = nS_{2,i} \quad \frac{n}{n \quad m_i} S_{1,i}^2 \qquad (2.3)$$

is an approximation to $_i$ and the relation (2.2) becomes

$$\hat{z}_i = z_i \quad \frac{n}{\delta_{1,i} \pm \sqrt{\frac{n \quad m_i}{m_i}\left(n\delta_{2,i} \quad \delta_{1,i}^2 \quad f_i\right)}}$$

$$= z_i \quad \frac{n}{\delta_{1,i} \pm \sqrt{\frac{n \quad m_i}{m_i} \quad n\delta_{2,i} \quad \delta_{1,i}^2 \quad nS_{2,i} + \frac{n}{n \quad m_i}S_{1,i}^2}}.$$

(2.4)

Let $z_1^{(0)},\cdots,z_k^{(0)}$ be initial approximations to the zeros $\lambda_1,\cdots,\lambda_k$ of *P*. Based on the last relation (2.4) we can construct the following iterative method of Laguerre's type for finding multiple zeros of a polynomial,

$$z_i^{(r+1)} = z_i^{(r)} \quad \frac{n}{\left[\delta_{1,i}^{(r)} + \frac{n \quad m_i}{m_i} \quad n\delta_{2,i}^{(r)} \quad \left(\delta_{1,i}^{(r)}\right)^2 \quad f_i^{(r)}\right]^{1/2}} \left(i \in I_k\right),$$

(2.5)

where the index $r = 0,1,\cdots$ is related to the *r*-th iterative step. If all the zeros of *P* are simple $(m_1 = m_2 = \cdots = m_n = 1)$, then the iterative method (2.5) reduces to the Laguerre-like simultaneous method in [1]. In [5, M. Petrović, L. Rancić, D. Milosević, Laguerre-like methods for the simultaneous approximation of polynomial multiple zeros, Yugoslav Journal of Operational Research, 16 (2006), Number 1, 31-44] is proved that if we use the already calculated approximations in the current iteration (Gauss-Seidel approach or serial mode), we obtain the Laguerre-like single-step method

$$z_i^{(r+1)} = z_i^{(k)} \quad \frac{n}{\left[\delta_{1,i}^{(r)} + \frac{n \quad m_i}{m_i} \quad n\delta_{2,i}^{(r)} \quad \left(\delta_{1,i}^{r}\right)^2 \quad \hat{f}_i^{(r)}\right]^{1/2}},$$

(2.6)

where

$$\hat{f}_i = n\left[\sum_{j=1}^{i\ 1} \frac{m_j}{(z_i \quad \hat{z}_j)^2} + \sum_{j=i+1}^{k} \frac{m_j}{(z_i \quad z_j)^2}\right] \quad \frac{n}{n \quad m_i}\left[\sum_{j=1}^{i\ 1} \frac{m_j}{z_i \quad \hat{z}_j} + \sum_{j=i+1}^{k} \frac{m_j}{z_i \quad z_j}\right]^2$$

with order of convergence increased by one.

3. SIMULTANEOUS METHODS FOR FINDING SIMPLE ZEROS

First, Durand in [2] and later Kerner in [3] indipendently proposed Durand-Kerner method also known as Weiestrass method

$$\hat{z}_i = z_i - W_i \qquad (3.1)$$

where

$$W_i = \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n}(z_i - z_j)} \qquad (3.2)$$

of the second order for the simultaneous finding of simple zeros of a polynomial *P*.

In this section we shall present iterative methods of Weierstrass' type for the simultaneous inclusion of simple zeros of a polynomial where the improved convergence is attained by using suitable corrections.

Let,

$$N_i = \frac{P(z_i)}{P'(z_i)}, \quad H_i = \frac{P(z_i)}{P'(z_i) - \frac{P(z_i)P''(z_i)}{2P'(z_i)}}$$

be Newton's and Halley's corrections appearing in the well-known iterative formulas $\hat{z}_i = z_i - N_i$ (Newton's method), $\hat{z}_i = z_i - H_i$ (Halley's method) of the second and third order, respectively.

Let be $z_1, z_2, \cdots, z_n$ the approximations to the zeros $\lambda_1, \cdots, \lambda_n$, of a monic polynomial of order *n*. Using the improved approximations $c_j = z_j - N_j$ or $c_j = z_j - H_j$ defined as the modified Weierstrass function

$$\tilde{W}_i(z) = \frac{P(z)}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n}(z - c_j)}. \qquad (3.3)$$

We don't stop here with the modifications done by M.S.Petrović and L.D.Petrović in [4]. The modification we propose follows an idea borrowed from numerical linear algebra, where it leads from Jacobi's method to Gauss-Seidel's. The idea is to use at every moment the latest computed components of the approximate solution vector in order to compute the next component, rather than using the "old" approximate solution vector to compute the entire "new" vector. The Gauss-Seidel approach or serial mode is applied in different methods to accelerate the convergence speed, for example in the Weierstrass method in [6], in the Laguerre method in [15] in the Laguerre-like method in [5] and in the interval method of Weierstrass' type in [14]. In this paper we apply this approach in the modified Weierstrass function (3.3) [4] achieving a new modification of the Weierstrass function

$$\vec{W}_i(z_i^{(r)}) = \frac{P(z_i^{(r)})}{\prod\limits_{j=1}^{i-1}\left(z_i^{(r)} - c_j^{(r+1)}\right)\prod\limits_{j=i+1}^{n}\left(z_i^{(r)} - c_j^{(r)}\right)},$$

$$i = 1, \cdots, n. \qquad (3.4)$$

All applications are based on the fact that the rational function W (or $\tilde{W}$, $\vec{W}$) has the same zeros as the polynomial *P*. We emphasize that the use of corrections is justified only when its evaluations can be performed by the already calculated quantities. In this way the order of convergence is increaced using negligible number of numerical operations giving a higher computation efficiency of the stated method.

Applying the new modified Weierstrass function $\vec{W}$ (3.4) proposed by us, instead of $P(z)$ in the Newton method, gives the new method

$$\hat{z}_i = z_i - \frac{\vec{W}(z_i)}{\vec{W}_i'(z_i)} \qquad (3.5)$$

which has order of convergenve five when we use Newton's corrections and six when we use Halley's corrections.

To construct other new methods with accelerated convergence we can apply the new modified Weierstrass function $\vec{W}$ in Ehrlich-Aberth method [13] and Ostrowski method [16], both of the third order for simple roots.

Finally, a few words about iterative methods with a known multiplicity. Most of the papers treating such methods begin with the phrase "Let α be a root of P with the given multiplicity m, ...," with no information how to provide the exact m.

In the following we shall consider the case when the order of multiplicity is not known. The idea most frequently used is to deflate all multiple

roots into simple ones. So, if α is a multiple root of a polynomial P, then α is a simple root of the ratio P/P'.
We refer to the following corollary [7, 8].

Corollary. Assume that $P(z)$ has n roots and among them there are k distinct roots, each denoted by λ_i with multiplicity m_i for $i=1,\cdots,k$, respectively. Then, $P(z)$ and its first derivative, $P'(z)$, have only one greatest common divisor (GCD)

$$P_c(z) = \prod_{i=1}^{k} (z - \lambda_i)^{m_i - 1}, \quad (3.6)$$

such that

$$P(z) = P_c(z)P_0(z) \text{ and } P'(z) = P_c(z)P_1(z), \quad (3.7)$$

where $P_0(z)$ has exactly the same k distinct roots, λ_i, as those of $P(z)$, which are all simple roots. The multiplicity of any root, λ_i, can be determined by

$$m_i = \frac{P_1(\lambda_i)}{P_0'(\lambda_i)}, \quad \text{for } i=1,\cdots,k. \quad (3.8)$$

Algorithm for finding simultaneously the polynomial roots with the respective multiplicities. Using the notations of the corollary we construct the following pseudocode.

Step 1. Compute $P'(z)$ of degree $n - 1$.

Step 2. Find the GCD $d_{gc}(z)$ of $P(z)$ and $P'(z)$ using the Euclidean algorithm.

Step 3. Compute $q_p(z) = P(z)/d_{gc}(z)$ and $q_g(z) = P'(z)/d_{gc}(z)$.

Step 4. Employ the simultaneous method (3.5) to determine all the k roots λ_i, distinct and simple, of $q_p(z)$.

Step 5. The multiplicities $m_i = 1$ for $i=1,\cdots,k$ if the GCD $d_{gc}(z)$ is a constant (polynomial). Otherwise, calculate the multiplicities $m_i = q_g(\lambda_i)/q_p'(\lambda_i)$.

Step 6. Output the k roots λ_i with their multiplicities m_i.

Two computational aspects of this algorithm are considered. Step 2 involves algebraic operations to search for the GCD of two polynomials. The step has computational complexity $O(n^2)$ with Euclidean algorithm, which is the extension to polynomials of the Euclidean algorithm for obtaining the GCD of two positive integers, or $O(n\log^2 n)$ with a fast version of the algorithm. Step 4 uses a simultaneous method to find simple roots of polynomials. The efficiency of this method that reveals its computational complexity is obtained by measuring the order of convergence. The order of convergence of our method is five (using the Newton's corrections) and six (using the Halley's corrections), and as we are using a simultaneous method for determining k simple roots, this algorithm finds simultaneously even their multiplicities. These are the reasons that lead us to try to implement this algorithm to a parallel computer in a further work.

4. NUMERICAL EXPERIMENTS

In this section we report on numerical experiments using Durand-Kerner method [1, 2] (DK) (3.1), Ehrlich-Aberth method [13] (EA), Durand-Kerner method with Newton's correction(DKN) (3.3) and Durand-Kerner method with Newton's correction and Gauss-Seidel approach (DKNGS).

In our example we took a cubic polynomial having three simple roots $x_1 = 2$, $x_2 = -1$, $x_3 = 4$, $P_3(x) = (x-2)(x+1)(x-4)$.

Here we started with initial approximations $x^{(0)} = (1.5; -1.5; 5)$ and the approximations achieved for the same error (10^{-4}) are given in Table 1. The results are obtained using Matlab 7.3.0.

Method	DK	EA	DKN	DKNGS
Order of convergenve	2	3	3	5
Number of iterations	5	3	3	2

Table 1

We have performed a lot of numerical experiments and found that the methods DKN and DKNGS demostrate very fast convercence even for crude initial approximations.

The convergence behaviour and numerical characteristics of the method (3.5) and the methods constructed by applying $\vec{W}$ to Ehrlich-Aberth method [13] and Ostrowski method [16] and their implementation in a parallel computer will be considered in the forthcoming work.

5. CONCLUSIONS

To find the multiple roots of polynomials simultaneously when the multiplicity is known, is a solved problem and as discussed in Section 2 the convergence order is at least four. The case when the multiplicity is not known is more difficult. We have presented an algorithm that deflates all multiple roots into simple ones and computes their multiplicities without using higher derivatives evaluations, compared with the formula proposed by Traub [9], Laguanelle and modified Laguanelle formula [10]. Then in this algorithm we have implemented the new method (3.5) with higher order of covergence compared with those for multiple roots and eliminates the evaluation of higher derivatives in the intermediate steps.

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