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FIXED POINT THEOREMS FOR FUZZY CONTRACTIVE MAPPINGS
DISA TEOREMA PËR PIKAT FIKSE TË FUNKSIONEVE FUZZY KONTRAKTIVE

[a]ARBEN ISUFATI, [b]ELIDA HOXHA
[a]Departamenti i Matematikës dhe Informatikës, Fakulteti i Shkencave te Natyrës, Universiteti i Gjirokastrës
[b]Departamenti i Matematikes, Universiteti i Tiranes, Tiranë
Email: benisufati@yahoo.com

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PERMBLEDHJE
Studimi i pikave fikse të funksioneve fuzzy nën kushte kontraktive ose lokalisht kontraktive lidhur me distancën d_∞ është i përdorshëm në llogaritjen e dimensioneve të Hansdorffit. Këto dimensione ndihmojnë studjuesit të kuptojnë hapësirat ε^∞ të cilat janë të përdorshme në fizikën e energjive shumë të larta. Qëllimi i këtij punimi është të prezantojë disa teorema të reja për pikat fikse të funksioneve fuzzy nën kushte kontraktive. Rrezultati kryesor: Le të jetë (X,d) hapësirë e plotë metrike. Atëherë përftojmë hapësirën fractale $(C(X),d_H)$ dhe hapësirën fractale fuzzy $(\mathcal{CB}(X),d_\infty)$. Në qoftë se funksioni fuzzy $T:\mathcal{CB}(X)\to\mathcal{CB}(X)$ kënaq disa kushte kontraktiviteti shtesë, në raport me rezultatet e deritanishme atëherë ky funksion ka një pikë fikse. Distanca d_∞ si dhe përafrimi që presupozon dobësimin e kushteve të kontraktivitetit të këtyre funksioneve në atë shkallë qëe të sigurojë ekzistencën e pikave fikse përfaqësojnë metodën e ndjekur për të arritur në përfundimin e kërkuar. Krahas fushave të mësipërme të zbatimit të pikave fikse fuzzy kjo teori gjen zbatim edhe në sistemet dinamike, teorinë e lojrave fuzzy etj.

SUMMARY
The study of fixed points of fuzzy set-valued mappings under contractive and locally contractive conditions related to the d_∞ - metric is useful for computing Hausdorff dimensions. These dimensions help us to understand ε^∞ -spaces which are used in high energy physics. The aim of this work is to present some new fixed point theorems for fuzzy set-valued mappings under contractive conditions. Let (X,d) be a complete metric space, then we get the fractal space $(C(X),d_H)$ and the fuzzy fractal space $(\mathcal{CB}(X),d_\infty)$. If the fuzzy mappings $T:\mathcal{CB}(X)\to\mathcal{CB}(X)$ satisfying additional conditions then these mappings have a fixed point. Using the properties of the distance d_∞ induced by the Hansdorff distance of the family of fuzzy sets several fixed point and common fixed point are obtained. The methods of successive approximations, used to approximate the fixed points. Fuzzy fixed point theory can be used in dynamical systems, fuzzy game theory, multi-valued fractals etc. Also, our results are useful in geometric problems arising in high energy physics.
Key words: Contractive mappings; d_∞ -metric; Fuzzy set.

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by Zadeh [17]. On subspaces of fuzzy sets are used many metrics, where most frequently have been studied the d_∞ -metric[5,6,7,13,14,16] induced by Hausdorff metric. Fixed point theorems for fuzzy set-valued mappings have been studied by Heilpern [4] who introduced the concept of fuzzy

contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces. Qiu and Shu [14] established the $\mathcal{CB}(X)$, the class of fuzzy sets with nonempty bounded closed α-cut sets equipped with d_∞-metric and proved common fixed point theorems for a sequence of fuzzy-self mappings in this.. Also Qiu and Shu [13] considered the $\mathcal{C}(X)$ the class of fuzzy sets with nonempty compact α-cut sets equipped with d_∞-metric. Common fixed point theorems for fuzzy self-mappings in $\mathcal{C}(X)$ under Φ-contraction conditions are proved. It is known that all compact sets are bounded closed sets in a general metric space and the converse is not always true. So in general the $\mathcal{C}(X)$ is proper subset of $\mathcal{CB}(X)$.
As corollary of results in [14] we have the following theorem which is the fuzzy version of well known Nadler theorem:

Theorem 1.1 Let (X,d) be a complete metric space and $\mathcal{CB}(X)$ the class of fuzzy sets with nonempty closed bounded α-cut sets, equipped with the supremum metric d_∞, $F:\mathcal{CB}(X)\to\mathcal{CB}(X)$ be a fuzzy self-mapping satisfying

$$d_\infty(F(\mu),F(\eta))\le q d_\infty(\mu,\eta)$$

where $0\le q<1$, for all $\mu,\eta\in\mathcal{CB}(X)$. Then there exists a $\mu_*\in\mathcal{CB}(X)$ such that $\mu_*\subseteq F(\mu_*)$.
In this paper we prove common fixed point theorem for a sequence of fuzzy self-mappings in $\mathcal{CB}(X)$ under φ-contraction condition using a new technique of proofs. Also a new class of fuzzy self-mappings on $\mathcal{CB}(X)$ satisfying a rational inequality is considered. Our theorems generalize and improve some recent results in literature.
Through this paper, we shall use the following notations which have been recoded from [14].
Let (X,d) be a metric space and let $CB(X)$ be the set of all bounded closed subsets of X. The Hausdorff metric is defined as:

$$H(A,B)=\max\{\sup_{x\in B}\inf_{y\in A}d(x,y),\sup_{x\in A}\inf_{y\in B}d(x,y)\}$$
$$=\max\{\rho(B,A),\rho(A,B)\}$$

where $A,B\in CB(X)$ and $\rho(A,B)=\sup_{x\in A}\inf_{y\in B}d(x,y)=\sup_{x\in A}d(x,B)$ is the Hausdorff separation of A from B.

Lemma 1[14] The metric space $(CB(X),H)$ is complete provided (X,d) is complete.
Let (X,d) be a metric space. A fuzzy set μ on X is defined by its membership function $\mu(x)$ which is a mapping from X into $[0,1]=I$. The α-cut of μ is

$$[\mu]^\alpha=\{x\in X:\mu(x)\ge\alpha\}$$

where $0<\alpha\le 1$, and the support $[\mu]^0$ of μ to be the closure of the union of $[\mu]^\alpha$ for $0<\alpha\le 1$.
The totality of fuzzy sets $\mu:X\to[0,1]=I$ which satisfy that for each $\alpha\in I$, the α-cut of μ is non-empty bounded closed in X is denoted by $\mathcal{CB}(X)$.
By $\mathcal{C}(X)$ is denoted the collection of all fuzzy sets $\mu:X\to[0,1]=I$ which satisfy that for each $\alpha\in I$, the α-cut of μ is non-empty compact in X. It is easy to see that $\mathcal{C}(X)$ is proper subset of $\mathcal{CB}(X)$.
Let $\mu_1,\mu_2\in\mathcal{CB}(X)$. Then μ_1 is said to be included in μ_2, denoted by $\mu_1\subseteq\mu_2$, if and only if $\mu_1(x)\le\mu_2(x)$ for each $x\in X$. Thus we have that $\mu_1\subseteq\mu_2$ if and only if $[\mu_1]^\alpha\subseteq[\mu_2]^\alpha$ for all $\alpha\in I$.
The d_∞-metric is induced by the Hausdorff metric is defined as

$$d_\infty(\mu_1,\mu_2)=\max\{\rho_\infty(\mu_1,\mu_2),\rho_\infty(\mu_2,\mu_1)\}$$

where $\mu_1,\mu_2\in\mathcal{CB}(X)$ and

$\rho_\infty(\mu_1,\mu_2)=\sup_{0\le\alpha\le1}\rho([\mu_1]^\alpha,[\mu_2]^\alpha)$ is the Hausdorff separation of μ_1 from μ_2.
Let $\{\mu_n\}$ be a sequence in $\mathcal{CB}(X)$. It follows from the definition of d_∞ that μ_n converges with respect to the d_∞-metric if and only if $[\mu_n]^\alpha$ converges uniformly in $\alpha\in I$ with respect to the Hausdorff metric.

Definition 1 [14] Let (X,d) and (Y,ρ) be two metric spaces. A mapping F is said to be a fuzzy mappings if and only if F is mapping from the space $\mathscr{CB}(X)$ into $\mathscr{CB}(Y)$, i.e., $F(\mu)\in\mathscr{CB}(Y)$ for each $\mu\in\mathscr{CB}(X)$. $\mu_0\in\mathscr{CB}(X)$ is said to be a fixed point of a fuzzy self-mapping F of $\mathscr{CB}(X)$ if and only if $\mu_0\subseteq F(\mu_0)$.

Lemma 2[14] Let $\mu_1,\mu_2,\mu_3\in\mathscr{CB}(X)$. Then we have

(i) $\rho_\infty(\mu_1,\mu_2)=0$ if and only if $\mu_1\subseteq\mu_2$

(ii) $d_\infty(\mu_1,\mu_2)=0$ if and only if $\mu_1=\mu_2$

(iii) if $\mu_1\subseteq\mu_2$ then $\rho_\infty(\mu_1,\mu_3)\le d_\infty(\mu_2,\mu_3)$

(iv) $\rho_\infty(\mu_1,\mu_3)\le d_\infty(\mu_1,\mu_2)+\rho_\infty(\mu_2,\mu_3)$

Theorem 1[14] The metric space $(\mathscr{CB}(X),d_\infty)$ is complete provided (X,d) is complete.

Theorem 3 [14] Let (X,d) be a metric space and $\mu_1,\mu_2\in\mathscr{CB}(X)$. Then

(i) for any $\varepsilon>0$ and any $\mu_3\in\mathscr{CB}(X)$ satisfying $\mu_3\subseteq\mu_1$, there exists a $\mu_4\in\mathscr{CB}(X)$ such that $\mu_4\subseteq\mu_2$ and

$$d_\infty(\mu_3,\mu_4)\le d_\infty(\mu_1,\mu_2)+\varepsilon$$

(ii) for any $\beta>1$ and any $\mu_3\in\mathscr{CB}(X)$ satisfying $\mu_3\subseteq\mu_1$, there exists a $\mu_4\in\mathscr{CB}(X)$ such that $\mu_4\subseteq\mu_2$ and

$$d_\infty(\mu_3,\mu_4)\le\beta d_\infty(\mu_1,\mu_2)$$

Theorem 4[14] Let (X,d) be a complete metric space and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$. If there exists a constant q, $0<q<1$, such that for each $\mu,\eta\in\mathscr{CB}(X)$

$$d_\infty(F_i(\mu),F_j(\eta))\le q\max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F_i(\mu)),\rho_\infty(\eta,F_j(\eta)),$$
$$[\rho_\infty(\mu,F_j(\eta))+\rho_\infty(\eta,F_i(\mu))]/2\}$$

then there exists a $\mu_*\in\mathscr{CB}(X)$ such that $\mu_*\subseteq F_i(\mu_*)$, for all $i\in Z^+$.

Lemma 3[2] Let $\varphi:R^+\to R^+$ be a non-decreasing function satisfying the following conditions: φ is continuous from the right and $\sum_{i=0}^{\infty}\varphi^i(t)<\infty$ (φ^i denote the i-th iterative function of φ). Then $\varphi(t)<t$.

3. COMMON FIXED POINT THEOREMS

Throughout this section φ be a function satisfying the conditions of Lemma 3.

Theorem 3.1 Let (X,d) be a complete metric space and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ such that

$$d_\infty(F_i(\mu),F_j(\eta))\le\varphi(m(\mu,\eta))\quad\text{for all }\mu,\eta\in\mathscr{CB}(X)\qquad(1)$$

where the strict inequality holds if $m(\mu,\eta)\ne0$ and

$$m(\mu,\eta)=\max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F_i(\mu)),\rho_\infty(\eta,F_j(\eta)),[\rho_\infty(\mu,F_j(\eta))+\rho_\infty(\eta,F_i(\mu))]/2\}$$

Then there exists a $\mu_*\in\mathscr{CB}(X)$ such that $\mu_*\subseteq F_i(\mu_*)$, for all $i\in Z^+$.

Proof. Let $\mu_0\in\mathscr{CB}(X)$ and $\mu_1\subseteq F_1(\mu_0)$. We may assume that $m(\mu_0,\mu_1)\ne0$, for otherwise $\rho_\infty(\mu_0,F_1(\mu_0))\le m(\mu_0,\mu_1)=0$ and by Lemma 2 $\mu_0\subseteq F_1(\mu_0)$ and μ_0 is the fixed point of F_1. Similarly, it can be shown that μ_0 is the fixed point of F_i and so μ_0 is the fixed point of $\{F_i\}_{i=1}^{\infty}$.

From (1) we have

$$d_\infty(F_1(\mu_0),F_2(\mu_1))<\varphi(m(\mu_0,\mu_1)).$$

So we may choose $\varepsilon_1>0$ with

$$d_\infty(F_1(\mu_0),F_2(\mu_1))+\varepsilon_1 \le \varphi(m(\mu_0,\mu_1)) . \quad (2)$$

By Theorem 3 we can find $\mu_2 \in \mathscr{CB}(X)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and

$$d_\infty(\mu_1,\mu_2) \le d_\infty(F_1(\mu_0),F_2(\mu_1))+\varepsilon_1 \qquad (3)$$

From the above two inequalities we get

$$\begin{aligned} d_\infty(\mu_1,\mu_2) &\le \varphi(m(\mu_0,\mu_1)) \\ &\le \varphi(\max\{d_\infty(\mu_0,\mu_1),\rho_\infty(\mu_0,F_1(\mu_0)),\rho_\infty(\mu_1,F_2(\mu_1)), \\ &\quad [\rho_\infty(\mu_0,F_2(\mu_1))+\rho_\infty(\mu_1,F_1(\mu_0))]/2\}) \\ &\le \varphi(\max\{d_\infty(\mu_0,\mu_1),d_\infty(\mu_1,\mu_2),[d_\infty(\mu_0,\mu_1)+d_\infty(\mu_1,\mu_2)]/2\}) \end{aligned} \qquad (4)$$

Since $\rho_\infty(\mu_0,F_1(\mu_0)) \le d_\infty(\mu_0,\mu_1)+\rho_\infty(\mu_1,F_1(\mu_0))$ and $\rho_\infty(\mu_1,F_1(\mu_0))=0$, we have $\rho_\infty(\mu_0,F_1(\mu_0)) \le d_\infty(\mu_0,\mu_1)$. Also $\rho_\infty(\mu_1,F_2(\mu_1)) \le d_\infty(\mu_1,\mu_2)$, since $\rho_\infty(\mu_1,F_2(\mu_1)) \le d_\infty(\mu_1,\mu_2)+\rho_\infty(\mu_2,F_1(\mu_1))$ and $\rho_\infty(\mu_2,F_1(\mu_1))=0$.

If $\max\{d_\infty(\mu_0,\mu_1),d_\infty(\mu_1,\mu_2),[d_\infty(\mu_0,\mu_1)+d_\infty(\mu_1,\mu_2)]/2\}=d_\infty(\mu_1,\mu_2)$ then

$$d_\infty(\mu_1,\mu_2) \le \varphi(d_\infty(\mu_0,\mu_1)) < d_\infty(\mu_1,\mu_2) \qquad (5)$$

which is a contradiction.

Thus $\max\{d_\infty(\mu_0,\mu_1),d_\infty(\mu_1,\mu_2),[d_\infty(\mu_0,\mu_1)+d_\infty(\mu_1,\mu_2)]/2\}=d_\infty(\mu_0,\mu_1)$ and so

$$d_\infty(\mu_1,\mu_2) \le \varphi(d_\infty(\mu_0,\mu_1)) \qquad (6)$$

By induction, we produce a sequence $\{\mu_n\}$ of points of $\mathscr{CB}(X)$ such that

$$\begin{cases} \mu_{n+1} \subseteq F_{n+1}(\mu_n), \ n=0,1,2,\ldots; \\ d_\infty(\mu_n,\mu_{n+1}) \le \varphi(d_\infty(\mu_{n-1},\mu_n)). \end{cases} \qquad (7)$$

From the above inequality and since φ is non-decreasing, we have

$$d_\infty(\mu_n,\mu_{n+1}) \le \varphi(d_\infty(\mu_{n-1},\mu_n)) \le \cdots \le \varphi^n(d_\infty(\mu_0,\mu_1)) \qquad (8)$$

Furthermore for arbitrary positive integers m and k, we have

$$d_\infty(\mu_{k+m},\mu_k) \le \sum_{i=k}^{k+m-1} d_\infty(\mu_i,\mu_{i+1}) \le \sum_{i=k}^{k+m-1} \varphi^i(d_\infty(\mu_0,\mu_1)) \qquad (9)$$

It follows that $\{\mu_n\}$ is a Cauchy sequence in $\mathscr{CB}(X)$. Since (X,d) is complete, by Theorem 1, $(\mathscr{CB}(X),d_\infty)$ is complete. Let $\mu_n \to \mu_* \in \mathscr{CB}(X)$. Next we show that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in Z^+$.

For arbitrary i and j, $i \ne j$, by Lemma 2 we have

$$\begin{aligned} \rho_\infty(\mu_*,F_i(\mu_*)) &\le d_\infty(\mu_*,\mu_j)+d_\infty(F_j(\mu_{j-1}),F_i(\mu_*)) \\ &\le d_\infty(\mu_*,\mu_j)+\varphi(\max\{d_\infty(\mu_{j-1},\mu_*),\rho_\infty(\mu_{j-1},F_j(\mu_{j-1})),\rho_\infty(\mu_*,F_i(\mu_*)), \\ &\quad [\rho_\infty(\mu_{j-1},F_i(\mu_*))+\rho_\infty(\mu_*,F_j(\mu_{j-1}))]/2\}) \\ &\le d_\infty(\mu_*,\mu_j)+\varphi(\max\{d_\infty(\mu_{j-1},\mu_*),d_\infty(\mu_{j-1},\mu_j),\rho_\infty(\mu_*,F_i(\mu_*)), \\ &\quad [d_\infty(\mu_j,\mu_*)+d_\infty(\mu_{j-1},\mu_*)+\rho_\infty(\mu_*,F_i(\mu_*))]/2\}) \end{aligned} \qquad (10)$$

For $j \to \infty$, in inequality (10) and using the continuity of φ, we have

$$\rho_\infty(\mu_*,F_i(\mu_*)) \le \varphi(\rho_\infty(\mu_*,F_i(\mu_*))) \qquad (11)$$

which implies $\rho_\infty(\mu_*,F_i(\mu_*))=0$. By Lemma 2, it follows that $\mu_* \subseteq F_i(\mu_*)$.

If in Theorem 3.1 we choose $\varphi(t)=kt$, where $k \in [0,1)$ is a constant, we obtain the following corollary

Corollary 3.2 [14, Theorem 4] Let (X,d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ such that

$$d_\infty(F_i(\mu),F_j(\eta)) \le k\max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F_i(\mu)),\rho_\infty(\eta,F_j(\eta)), [\rho_\infty(\mu,F_j(\eta))+\rho_\infty(\eta,F_i(\mu))]/2\} \quad (12)$$

for all $\mu,\eta \in \mathscr{CB}(X)$. Then there exists a $\mu_* \in \mathscr{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$, for all $i \in Z^+$.

Corollary 3.3 Let (X,d) be a complete metric space and let F be a fuzzy self-mapping of $\mathscr{CB}(X)$ such that

$$d_\infty(F(\mu),F(\eta)) \le \varphi(m(\mu,\eta)) \quad \text{for all } \mu,\eta \in \mathscr{CB}(X) \quad (13)$$

where the strict inequality holds if $m(\mu,\eta) \ne 0$ and

$$m(\mu,\eta) = \max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F(\mu)),\rho_\infty(\eta,F(\eta)),[\rho_\infty(\mu,F(\eta))+\rho_\infty(\eta,F(\mu))]/2\}.$$

Then there exists a $\mu_* \in \mathscr{CB}(X)$ such that $\mu_* \subseteq F(\mu_*)$.

Proof. As $F_i = F$, for $i = 1,2,\ldots$ in Theorem 3.1 we have the Corollary 3.3 .

Let $q \in [0,1/2)$. From

$$q\max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F(\mu)),\rho_\infty(\eta,F(\eta)),\rho_\infty(\mu,F(\eta)),\rho_\infty(\eta,F(\mu))\} \le 2qm(\mu,\eta)$$

and Corollary 3.2, we deduce the following corollaries.

Corollary 3.4 Let (X,d) be a complete metric space and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ such that

$$d_\infty(F_i(\mu),F_j(\eta)) \le k\max\{d_\infty(\mu,\eta),\rho_\infty(\mu,F_i(\mu)),\rho_\infty(\eta,F_j(\eta)), \rho_\infty(\mu,F_j(\eta)),\rho_\infty(\eta,F_i(\mu))\} \quad (14)$$

for all $\mu,\eta \in \mathscr{CB}(X)$. Then there exists a $\mu_* \in \mathscr{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$, for all $i \in Z^+$.

From Corollary 3.4, we deduce the following corollaries.

Corollary 3.5 Let (X,d) be a complete metric space and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathscr{CB}(X)$ such that

$$d_\infty(F_i(\mu),F_j(\eta)) \le a_1 d_\infty(\mu,\eta) + a_2\rho_\infty(\mu,F_i(\mu)) + a_3\rho_\infty(\eta,F_j(\eta)) + a_4\rho_\infty(\mu,F_j(\eta)) + a_5\rho_\infty(\eta,F_i(\mu)) \quad (15)$$

for all $\mu,\eta \in \mathscr{CB}(X)$, where a_1,a_2,a_3,a_4,a_5 are nonnegative real numbers with $\sum_{i=1}^{5} a_i < 1$ and $a_4 \ge a_5$.

Then there exists a $\mu_* \in \mathscr{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$, for all $i \in Z^+$.

Corollary 3.6 Let (X,d) be a complete metric space and let F be a fuzzy self-mapping of $\mathscr{CB}(X)$ such that

$$d_\infty(F(\mu),F(\eta)) \le \lambda d_\infty(\mu,\eta) \quad (16)$$

for all $\mu,\eta \in \mathscr{CB}(X)$, where $\lambda \in [0,1)$. Then there exists a $\mu_* \in \mathscr{CB}(X)$ such that $\mu_* \subseteq F(\mu_*)$.

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