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A RELATED FIXED POINT THEOREM FOR 4 MAPINGS ON 4 COMPLETE METRIC SPACES

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SUMMARY
We prove a related fixed point theorem for 4 mappings in 4 metric spaces using an implicit relation. This result extends, generalizes and unifies several of well-known fixed point theorems for contractive-type mappings on metric spaces. Two examples are obtained to illustrate the theorem. One of the conditions of the theorems is the continuity of three from four mappings. A counterexample shows that this condition is necessary. The extending of the well-known results for one, two and three metric spaces to four metric spaces and their generalization has been obtained by using the implicit relation. Corollaries of our theorem are considered. From each form of implicit relation is taken a respective corollary. In our theorem, the continuity of all mappings except one is necessary. The investigation of these results in case of an arbitrary number of m mappings to m complete metric spaces is an open problem for future consideration.
Key words: Cauchy sequence, complete metric space, fixed point, implicit relation.

1. INTRODUCTION

Many authors [1], [2], [4], [9] etc. have proved fixed point theorems on metric spaces for mappings satisfying implicit relation. In this paper, we will prove a new fixed point theorem for four mappings on four metric spaces, three of mappings must be continuous.

In [6], the following theorem is proved.

Theorem 1.1 (Jain et al [6]) Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities

$$d(RSTx,RSTx') \leq c\max\{d(x,x'),d(x,RSTx),d(x',RSTx')\rho(Tx,Tx'),\sigma(STx,STx')\}$$
$$\rho(TRSy,TRSy') \leq c\max\{\rho(y,y'),\rho(y,TRSy),\rho(y',TRSy')\sigma(Sy,Sy'),d(RSy,RSy')\}$$
$$\sigma(STRz,STRz') \leq c\max\{\sigma(z,z'),\sigma(z,STRz),\sigma(z',STRz'),d(Rz,Rz'),\rho(TRz,TRz')\}$$

for all $x,x'\in X; y,y'\in Y$ and $z,z'\in Z$ where $0\leq c<1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu=v, Sv=w$ and $Rw=u$.

2. MAIN RESULTS

Firstly, we consider a new class of implicit relations which will give a more general character to the main Theorem 2.6.

Let $R_+ = [0,+\infty)$. We denote by Φ_7 the set of all functions with 7 variables $\varphi : R_+^7 \to R$ satisfying the properties:

(a). φ is upper semi-continuous in each coordinate variable $t_1, t_2, \ldots, t_7$

(b). If $\varphi(u,v,v,u,v_1,v_2,v_3) \le 0$ or $\varphi(u,v,u,v,v_1,v_2,v_3) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,v_3) \le 0$ for all $u,v,v_1,v_2,v_3 \ge 0$, then there exists a real constant $0 \le c < 1$ such that $u \le c\max\{v,v_1,v_2,v_3\}$.

Every such function will be called a Φ_7-function with constant c.

Example 2.1 The function $\varphi(t_1,t_2,\ldots,t_7) = t_1^p - c\max\{t_2^p,t_3^p,\ldots,t_7^p\}$, where $0 \le c < 1$ and $p \ge 0$, is Φ_7-function with constant c.

Proof. (a) is clear since φ is continuous. Suppose that $u,v,v_1,v_2,v_7 \ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,v_7) = u^p - c\max\{v^p,v^p,u^p,v_1^p,v_2^p,v_3^p\} =$$

$$= u^p - c\max\{u^p,v^p,v_1^p,v_2^p,v_3^p\} \le 0$$

If $u^p \ge \max\{v^p,v_1^p,v_2^p,v_3^p\}$, then $u^p \le c\max\{u^p,v^p,v_1^p,v_2^p,v_3^p\} = cu^p < u^p$, a contradiction. Therefore, $u^p \le c\max\{v^p,v_1^p,v_2^p,v_3^p\}$ and so $u \le c_1\max\{v,v_1,v_2,v_3\}$

where $c_1 = \sqrt[p]{c} < 1$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,v_3) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,v_3) \le 0$, then $u \le c_1\max\{v,v_1,v_2,v_3\}$.

The proof of (b) is completed.

Example 2.2 The function $\varphi(t_1,t_2,\ldots,t_7) = t_1 - (a_2t_2^p + a_3t_3^p + \ldots + a_7t_7^p)^{1/p}$, where $p > 0$ and $0 \le a_i, \sum_{i=2}^{7} a_i < 1$, $i = 2,3,\ldots,7$, is Φ_7-function with constant $c = \sum_{i=2}^{7} a_i$

Proof: (a) is clear since φ is continuous. Suppose that $u,v,v_1,v_2,v_3 \ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,v_3) = u - (a_2v^p + a_3v^p + a_4u^p + a_5v_1^p + a_6v_2^p + a_7v_3^p)^{1/p} \le 0$$

If $u^p \ge \max\{v^p,v_1^p,v_2^p,v_3^p\}$, then

$$u \le (a_2v^p + a_3v^p + a_4u^p + a_5v_1^p + a_6v_2^p + a_7v_3^p)^{1/p} \le (a_2u^p + a_3u^p + a_4u^p + a_5u^p + a_6u^p + a_7u^p)^{1/p} =$$

$$= [(a_2 + a_3 + a_4 + a_5 + a_6 + a_7)u^p]^{1/p} = (a_2 + a_3 + \ldots + a_7)^{1/p}u = cu < u,$$

a contradiction, where $c = (a_2 + a_3 + \ldots + a_7)^{1/p} < 1$. Therefore,

$u \le [(a_2 + \ldots + a_7)\max\{v^p,v_1^p,v_2^p,v_3^p\}]^{1/p} = c\max\{v,v_1,v_2,v_3\}$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,v_3) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,v_3) \le 0$, then $u \le c_1\max\{v,v_1,v_2,v_3\}$. The proof of (b) is completed.

We denote by $\cdot_6$ the set of all continuous functions with 6 variables $f : R_+^6 \to R$ satisfying the properties:

(a'). f is non decreasing in respect with each variable.

(b'). $f(t,t,\ldots,t) \le t$, $t \in R_+$

Every such function will be called a $\bullet_6$-function

Denote $I_6 = \{1,2,\ldots,6\}$. Some examples of $\bullet_6$-function are as follows:

1. $f(t_1,t_2,\ldots,t_6) = \max\{t_1,t_2,\ldots,t_6\}$

2. $f(t_1,t_2,\ldots,t_6) = [\max\{t_i t_j : i,j \in I_6\}]^{1/2}$

3. $f(t_1,t_2,\ldots,t_6) = [\max\{t_1t_2, t_2t_3,\ldots,t_5t_6\}]^{1/2}$

4. $f(t_1,t_2,\ldots,t_6) = [\max\{t_1^p, t_2^p,\ldots,t_6^p\}]^{1/p}$ $, p > 0$

5. $f(t_1,t_2,\ldots,t_6) = (a_1t_1^p + a_2t_2^p + \ldots + a_6t_6^p)^{1/p}$, where $p > 0$ and $0 \le a_i, \sum_{i=1}^{6} a_i \le 1$

The proof is done for the example 5 :

(a') It is obvious that the function f is non decreasing in respect with each variable

(b') We have: $f(t,t,\ldots,t) = (a_1t^p + a_2t^p + \ldots + a_6t^p)^{1/p} = [(a_1 + a_2 + \ldots + a_6)t^p]^{1/p} =$

$= (a_1 + a_2 + \ldots + a_6)^{1/p} t \le t,\ t \in R_+$. The proof of (b') is completed.

The following relationship between $\bullet_6$-functions and Φ_7-functions holds:

Lemma 2.3 *If* $f \in \bullet_6$ *and* $0 \le c < 1$*, then the function* $\varphi(t_1,t_2,\ldots,t_7) = t_1 - cf(t_2,t_3,\ldots,t_6,t_7)$ *is* Φ_7*-function with constant c*

Proof. (a) is clear since φ is continuous. Suppose that $u,v,v_1,v_2,v_3 \ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,v_3) = u - cf(v,v,u,v_1,v_2,v_3) \le 0 \qquad (*)$$

We have $u \le \max\{v,v_1,v_2,v_3\}$ since in contrary, (if $u > \max\{v,v_1,v_2,v_3\}$), by using the properties of f we get: $f(v,v,u,v_1,v_2,v_3) \le f(u,u,\ldots,u) \le u$ and by (*) it follows $u \le cu < u$, a contradiction. Therefore, after replacing the coordinates of the point (v,v,u,v_1,v_2,v_3) by $\max\{v,v_1,v_2,v_3\}$ and using the properties of f we get $u \le c\max\{v,v_1,v_2,v_3\}$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,v_3) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,v_3) \le 0$, then $u \le c\max\{v,v_1,v_2,v_3\}$. The proof of (b) is completed.

The above lemma gives us the possibility to establish other functions of type Φ_7:

Example 2.4 $\varphi(t_1,t_2,\ldots,t_7) = t_1 - c[\max\{t_2t_3, t_3t_4,\ldots,t_6t_7\}]^{1/2}$, where $0 \le c < 1$.

Example 2.5 $\varphi(t_1,t_2,\ldots,t_7) = t_1 - c\dfrac{t_2 + t_3 + \ldots + t_7}{6}$, where $0 \le c < 1$ etc.

Now, we prove the following theorem for *m* mappings on *m* metric spaces using an implicit relation.

Theorem 2.6 Let $(X,d), (Y,\rho), (Z,\sigma)$ and (W,τ) be complete metric spaces and $T: X \to Y$, $S: Y \to Z$, $R: Z \to W$ and $Q: W \to X$ be maps, at least three from which are continuous. If the following inequalities are satisfied:

$$\varphi_1\begin{pmatrix} d(QRSTx, QRSTx'), d(x,x'), d(x, QRSTx), d(x', QRSTx'), \\ \rho(Tx,Tx'), \sigma(STx, STx'), \tau(RSTx, RSTx') \end{pmatrix} \le 0 \qquad (1)$$

$$\varphi_2\begin{pmatrix} \rho(TQRSy, TQRSy'), \rho(y,y'), \rho(y, TQRSy), \rho(y', TQRSy'), \\ \sigma(Sy, Sy'), \tau(RSy, RSy'), d(QRSy, QRSy') \end{pmatrix} \le 0 \qquad (2)$$

$$\varphi_3\begin{pmatrix} \sigma(STQRz, STQRz'), \sigma(z,z'), \sigma(z, STQRz), \sigma(z', STQRz'), \\ \tau(Rz, Rz'), d(QRz, QRz'), \rho(TQRz, TQRz') \end{pmatrix} \le 0 \qquad (3)$$

$$\varphi_4\begin{pmatrix} \tau(RSTQw, RSTQw'), \tau(w,w'), \tau(w, RSTQw), \tau(w', RSTQw'), \\ d(Qw, Qw'), \rho(TQw, TQw'), \sigma(STQw, STQw') \end{pmatrix} \le 0 \qquad (4)$$

for all $x,x' \in X; y,y' \in Y; z,z' \in Z$ and $w,w' \in W$ where $\varphi_1,\varphi_2,\varphi_3,\varphi_4 \in \Phi_7$, then QRST has a unique fixed point α in X, TQRS has a unique fixed point β in Y, STQR has a unique fixed point γ in Z and RSTQ has a unique fixed point δ in W. Further, $T\alpha=\beta, S\beta=\gamma, R\gamma=\delta$ and $Q\delta=\alpha$.

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequences $(x_n),(y_n),(z_n)$ and (w_n) in X,Y,Z and W respectively as follows:

$$x_n = (QRST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n, w_n = Rz_n, n \in N$$

We prove that $(x_n),(y_n),(z_n)$ and (w_n) ere Cauchy sequences.

Denote

$$d_n = d(x_n, x_{n+1}), \rho_n = \rho(y_n, y_{n+1}), \sigma_n = \sigma(z_n, z_{n+1}), \tau_n = \tau(w_n, w_{n+1}), n \in N$$

We will assume that $x_n \ne x_{n+1}$, $y_n \ne y_{n+1}$, $z_n \ne z_{n+1}$ and $w_n \ne w_{n+1}$ for all *n*, otherwise if $x_n = x_{n+1}$ for some *n* , then $y_{n+1} = y_{n+2}, z_{n+1} = z_{n+2}$ and $w_{n+1} = w_{n+2}$ we can take $\alpha = x_n, \beta = y_{n+1}, \gamma = z_{n+1}, \delta = w_{n+1}$.

By the inequality (2), for $y = y_{n-1}$ and $y' = y_n$ we get:

$$\varphi_2\begin{pmatrix} \rho(TQRSy_{n-1}, TQRSy_n), \rho(y_{n-1}, y_n), \rho(y_{n-1}, TQRSy_{n-1}), \rho(y_n, TQRSy_n), \\ \sigma(Sy_{n-1}, Sy_n), \tau(RSy_{n-1}, RSy_n), d(QRSy_{n-1}, QRSy_n) \end{pmatrix} =$$

$$= \varphi_2\begin{pmatrix} \rho(y_n, y_{n+1}), \rho(y_{n-1}, y_n), \rho(y_{n-1}, y_n), \rho(y_n, y_{n+1}), \\ \sigma(z_{n-1}, z_n), \tau(w_{n-1}, w_n), d(x_{n-1}, x_n) \end{pmatrix} =$$

$$= \varphi_2(\rho_n, \rho_{n-1}, \rho_{n-1}, \rho_n, \sigma_{n-1}, \tau_{n-1}, d_{n-1}) \le 0$$

And from **(b),** we have $\rho_n \le c\max\{\rho_{n-1}, \sigma_{n-1}, \tau_{n-1}, d_{n-1}\}$ or

$$\rho_n \le c\max\{d_{n-1}, \rho_{n-1}, \sigma_{n-1}, \tau_{n-1}\} \qquad (5)$$

We have denoted by $c = \max\{c_1, c_2, c_3, c_4\}$ where c_i is the constant of Φ_7-function φ_i, $i = 1,2,3,4$.

By (3) for $z = z_{n-1}$ and $z' = z_n$ we get:

$$\varphi_3\begin{pmatrix} \sigma(STQRz_{n-1}, STQRz_n), \sigma(z_{n-1}, z_n), \sigma(z_{n-1}, STQRz_{n-1}), \sigma(z_n, STQRz_n), \\ \tau(Rz_{n-1}, Rz_n), d(QRz_{n-1}, QRz_n), \rho(TQRz_{n-1}, TQRz_n) \end{pmatrix} =$$

$$= \varphi_3\begin{pmatrix} \sigma(z_n, z_{n+1}), \sigma(z_{n-1}, z_n), \sigma(z_{n-1}, z_n), \sigma(z_n, z_{n+1}), \\ \tau(w_{n-1}, w_n), d(x_{n-1}, x_n), \rho(y_n, y_{n+1}) \end{pmatrix} =$$

$$= \varphi_3(\sigma_n, \sigma_{n-1}, \sigma_{n-1}, \sigma_n, \tau_{n-1}, d_{n-1}, \rho_n) \le 0$$

And from **(b),** we have $\sigma_n \le c\max\{\sigma_{n-1}, \tau_{n-1}, d_{n-1}, \rho_n\}$

By this inequality and by (5) it follows

$$\sigma_n \le c\max\{d_{n-1}, \rho_{n-1}, \sigma_{n-1}, \tau_{n-1}\} \qquad (6)$$

By (4) for $w = w_{n-1}$ and $w' = w_n$ we get:

$$\varphi_4\begin{pmatrix} \tau(RSTQw_{n-1}, RSTQw_n), \tau(w_{n-1}, w_n), \tau(w_{n-1}, RSTQw_{n-1}), \tau(w_n, RSTQw_n), \\ d(Qw_{n-1}, Qw_n), \rho(TQw_{n-1}, TQw_n), \sigma(STQw_{n-1}, STQw_n) \end{pmatrix} =$$

$$= \varphi_4\begin{pmatrix} \tau(w_n, w_{n+1}), \tau(w_{n-1}, w_n), \tau(w_{n-1}, w_n), \tau(w_n, w_{n+1}), \\ d(x_{n-1}, x_n), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}) \end{pmatrix} =$$

$$= \varphi_4(\tau_n, \tau_{n-1}, \tau_{n-1}, \tau_n, d_{n-1}, \rho_n, \sigma_n) \le 0$$

And from **(b),** we have $\tau_n \le c\max\{\tau_{n-1}, d_{n-1}, \rho_n, \sigma_n\}$

By this inequality and by (5) and (6) it follows

$$\tau_n \le c\max\{d_{n-1}, \rho_{n-1}, \sigma_{n-1}, \tau_{n-1}\} \qquad (7)$$

In similar way, by (1) for $x = x_{n-1}$ and $x' = x_n$ we get:

$$d_n \le c\max\{d_{n-1}, \rho_{n-1}, \sigma_{n-1}, \tau_{n-1}\} \qquad (8)$$

By the inequalities (5), (6), (7) and (8), using the mathematical induction, we get:

$$d(x_n, x_{n+1}) \le c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \tau(w_1, w_2)\} = c^{n-1}l$$

$$\rho(y_n, y_{n+1}) \le c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \tau(w_1, w_2)\} = c^{n-1}l$$

$$\sigma(z_n, z_{n+1}) \le c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \tau(w_1, w_2)\} = c^{n-1}l$$

$$\tau(w_n, w_{n+1}) \le c^{n-1}\max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \tau(w_1, w_2)\} = c^{n-1}l$$

where $l = \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \tau(w_1, w_2)\}$

Thus the sequences $(x_n), (y_n), (z_n)$ and (w_n) are Cauchy sequences. Since the metric spaces $(X,d), (Y,\rho), (Z,\sigma)$ and (w,τ) are complete metric spaces we have:

$$\lim_{n\to\infty} x_n = \alpha \in X,\ \lim_{n\to\infty} y_n = \beta \in Y,\ \lim_{n\to\infty} z_n = \gamma \in Z,\ \lim_{n\to\infty} w_n = \delta \in W.$$

Now suppose that T, S, R are continuous mappings, we have:

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n \Rightarrow T\alpha = \beta.$$

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n \Rightarrow S\beta = \gamma. \qquad (9)$$

$$\lim_{n\to\infty} Rz_n = \lim_{n\to\infty} w_n \Rightarrow R\gamma = \delta.$$

Later we will show that $Q\delta = \alpha$.

To prove that α is a fixed point of $QRST$.

Using the inequality (1), for $x = \alpha$ and $x' = x_{n-1}$, we obtain:

$$\varphi_1\begin{pmatrix} d(QRST\alpha, x_n), d(\alpha, x_{n-1}), d(\alpha, QRST\alpha), d(x_{n-1}, x_n), \\ \rho(T\alpha, Tx_{n-1}), \sigma(ST\alpha, STx_{n-1}), \tau(RST\alpha, RSTx_{n-1}) \end{pmatrix} \le 0$$

Letting n tend to infinity and using (a) and the continuality of T, S, R we have

$$\varphi_1\begin{pmatrix} d(QRST\alpha, \alpha), d(\alpha, \alpha), d(\alpha, QRST\alpha), d(\alpha, \alpha), \\ \rho(T\alpha, T\alpha), \sigma(ST\alpha, ST\alpha), \tau(RST\alpha, RST\alpha) \end{pmatrix} =$$

$$= \varphi_1(d(QRST\alpha, \alpha), 0, d(\alpha, QRST\alpha), 0, 0, 0, 0) \le 0$$

and from **(b)**, we have: $d(QRST\alpha, \alpha) \le c\max\{0,0,0,0\} = 0$

Thus $d(QRST\alpha, \alpha) = 0$ and so α is a fixed point of $QRST$

We now have

$$TQRS\beta = T(QRST\alpha) = T\alpha = \beta$$
$$STQR\gamma = S(TQRS\beta) = S\beta = \gamma$$
$$RSTQ\delta = R(STQR\gamma) = R\gamma = \delta$$

Hence β, γ, δ are fixed points of $TQRS$, $STQR, RSTQ$ respectively

We now prove the uniqueness of the fixed point α .

Suppose that $QRST$ has a second fixed point $\alpha' \neq \alpha$. Using the inequality (1) for $x = \alpha$ and $x' = \alpha'$ we have:

$$\varphi_1\begin{pmatrix} d(QRST\alpha, QRST\alpha'), d(\alpha,\alpha'), d(\alpha, QRST\alpha), d(\alpha', QRST\alpha'), \\ \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha'), \tau(RST\alpha, RST\alpha') \end{pmatrix} =$$
$$= \varphi_1\left(d(\alpha,\alpha'), d(\alpha,\alpha'), 0, 0, \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha'), \tau(RST\alpha, RST\alpha')\right) \leq 0$$

And from (b), we have:

$$d(\alpha,\alpha') \leq c\max\{\rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha'), \tau(RST\alpha, RST\alpha')\} \qquad (10)$$

In similar way, applying the inequality (2) for $y = T\alpha$ and $y' = T\alpha'$, by the property (b) of φ_2 and taking in consideration (10) we obtain:

$$\rho(T\alpha, T\alpha') \leq c\max\{\sigma(ST\alpha, ST\alpha'), \tau(RST\alpha, RST\alpha')\} \qquad (11)$$

Similarly, applying the inequality (3) for $z = ST\alpha$ and $z' = ST\alpha'$, we have

$$\sigma(ST\alpha, ST\alpha') \leq c\tau(RST\alpha, RST\alpha') \qquad (12)$$

Applying the inequality (4), for $w = RST\alpha$, $w' = RST\alpha'$ and using the property (b) of φ_4 and these inequalities $(10),(11),(12)$, we now have

$$\tau(RST\alpha, RST\alpha') \leq c\tau(RST\alpha, RST\alpha')$$

and so

$$\tau(RST\alpha, RST\alpha') = 0 \qquad (13)$$

Returning back and using $(13),(12),(11)$ we get: $d(\alpha,\alpha') = 0$

And so, $\alpha = \alpha'$, then the uniqueness of α is proved. In the same way it can be proved the uniqueness of β, γ and δ .

We finally prove that also we have $Q\delta = \alpha$. To do this, note that

$$Q\delta = Q(RSTQ\delta) = QRST(Q\delta)$$

and so, $Q\delta$ is a fixed point of $QRST$. Since α is the unique fixed point, it follows that $Q\delta = \alpha$. This completes the proof of the theorem.

***3.* Corollaries**

The next corollary follows from theorem 2.6 in the case $\varphi_i(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = t_1 - c_i f_i(t_2, t_3, t_4, t_5, t_6, t_7)$, where $f_i \in \bullet_6$ for i=1,2,3,4.

Corollary 3.1 Let $(X,d),(Y,\rho)$, (Z,σ) and (W,τ) be complete metric spaces and $T: X \to Y$, $S: Y \to Z$, $R: Z \to W$ and $Q: W \to X$ be maps, at least three from which are continuous. If the following inequalities are satisfied:

$$d(QRSTx, QRSTx') \leq f_1\{d(x, x'), d(x, QRSTx), d(x', QRSTx'),$$
$$\rho(Tx, Tx'), \sigma(STx, STx'), \tau(RSTx, RSTx')\}$$
$$\rho(TQRSy, TQRSy') \leq f_2\{\rho(y, y'), \rho(y, TQRSy), \rho(y', TQRSy'),$$
$$\sigma(Sy, Sy'), \tau(RSy, RSy'), d(QRSy, QRSy')\}$$
$$\sigma(STQRz, STQRz') \leq f_3\{\sigma(z, z'), \sigma(z, STQRz), \sigma(z', STQRz'),$$
$$\tau(Rz, Rz'), d(QRz, QRz'), \rho(TQRz, TQRz')\}$$
$$\tau(RSTQw, RSTQw') \leq f_4\{\tau(w, w'), \tau(w, RSTQw), \tau(w', RSTQw'),$$
$$d(Qw, Qw'), \rho(TQw, TQw'), \sigma(STQw, STQw')\}$$

for all $x,x' \in X; y,y' \in Y; z,z' \in Z$ and $w,w' \in W$ where $f_1,f_2,f_3,f_4 \in {\bullet}_6$, then QRST has a unique fixed point α in X, TQRS has a unique fixed point β in Y, STQR has a unique fixed point γ in Z and RSTQ has a unique fixed point δ in W. Further, $T\alpha = \beta, S\beta = \gamma, R\gamma = \delta$ and $Q\delta = \alpha$.

The next corollary follows from theorem 2.6 in the case $\phi_i(t_1,t_2,t_3,t_4,t_5,t_6,t_7) = t_1 - c_i \max\{t_2,t_3,t_4,t_5,t_6,t_7\}$, for i=1,2,3,4.

Corollary 3.2 Let (X,d), (Y,ρ), (Z,σ) and (W,τ) be complete metric spaces and $T: X \to Y$, $S: Y \to Z$, $R: Z \to W$ and $Q: W \to X$ be maps, at least three from which are continuous. If the following inequalities are satisfied:

$$d(QRSTx, QRSTx') \leq c_1 \max\{d(x, x'), d(x, QRSTx), d(x', QRSTx'),$$
$$\rho(Tx, Tx'), \sigma(STx, STx'), \tau(RSTx, RSTx')\} \qquad (1')$$
$$\rho(TQRSy, TQRSy') \leq c_2 \max\{\rho(y, y'), \rho(y, TQRSy), \rho(y', TQRSy'),$$
$$\sigma(Sy, Sy'), \tau(RSy, RSy'), d(QRSy, QRSy')\} \qquad (2')$$
$$\sigma(STQRz, STQRz') \leq c_3 \max\{\sigma(z, z'), \sigma(z, STQRz), \sigma(z', STQRz'),$$
$$\tau(Rz, Rz'), d(QRz, QRz'), \rho(TQRz, TQRz')\} \qquad (3')$$
$$\tau(RSTQw, RSTQw') \leq c_4 \max\{\tau(w, w'), \tau(w, RSTQw), \tau(w', RSTQw'),$$
$$d(Qw, Qw'), \rho(TQw, TQw'), \sigma(STQw, STQw')\} \qquad (4')$$

for all $x,x' \in X; y,y' \in Y; z,z' \in Z$ and $w,w' \in W$ where $f_1,f_2,f_3,f_4 \in {\bullet}_6$, then QRST has a unique fixed point α in X, TQRS has a unique fixed point β in Y, STQR has a unique fixed point γ in Z and RSTQ has a unique fixed point δ in W. Further, $T\alpha = \beta, S\beta = \gamma, R\gamma = \delta$ and $Q\delta = \alpha$.

Corollary 3.3 From the corollary 3.2 we obtain the theorem 1.1 (Jain et al [6]).

Proof. The proof follows by Corollary 3.2 in the case $W = X, \tau = d$, $w = x$, $w' = x'$ and the mapping Q as the identity mapping in X.

Corollary 3.4 From the corollary 3.3 we obtain the Fisher theorem [3]:

Let (X,d) and (Y,ρ) are complete metric spaces and $T: X \to Y$, $S: Y \to X$ be two maps, at least one of them being continuous. If for some $c \in [0,1)$ the following inequalities are satisfied:

$$d(STx, STx') \leq c\max\{d(x,x'), d(x,STx), d(x',STx'), \rho(Tx,Tx')\}$$
$$\rho(TSy, TSy') \leq c\max\{\rho(y,y'), \rho(y,TSy), \rho(y',TSy'), d(Sy,Sy')\}$$

for all $x,x'\in X;\ y,y'\in Y$, then ST has a unique fixed point $\alpha\in X$ and TS has a unique fixed point $\beta\in Y$. Moreover, $T\alpha=\beta$ and. $S\beta=\alpha$

Proof. The proof follows by Corollary 3.3 (Theorem 1.1) in the case $Z=X,\sigma=d$, $z=x$, $z'=x'$ and the mapping R as the identity mapping in X.

Corollary 3.5 From the corollary 3.4 we obtain the Rhoades theorem [11]:
Let (X,d) be a complete metric space and $T:X\to X$ a self map of X. If for some $c\in[0,1)$ we have

$$d(Tx,Tx')\le c\max\{d(x,x'),d(x,Tx),d(x',Tx')\}$$

for all $x,x'\in X$, then T has a unique fixed point α in X.

Proof. The proof follows by Corollary 3.4 in the case $Y=X,\rho=d$, $y=x$, $y'=x'$ and the mapping S as the identity mapping in X.

Remark: The theorems of Banach, Kannan, Bianchini, Reich follow from Corollary 3.5. For example, the proof of the Kannan theorem [7] follows from the fact that:

$$\max\{d(x,x'),d(x,Tx),d(x',Tx')\}\ge\frac{d(x,Tx)+d(x',Tx')}{2}$$, etc.

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