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A FIXED POINT THEOREM FOR m MAPINGS ON m COMPLETE FUZZY METRIC SPACES USING IMPLICIT RELATIONS

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SUMMARY

In this paper, we give some fixed point theorems for two, three and generally, for *m* mappings in m fuzzy metric spaces, $m-2$ of mappings must be continuous. These results extend, generalize, unify and fuzzyfy some of well-known fixed point theorems for contractive-type mappings in metric spaces for example the theorem of Nung, Jain et al., Popa, Telci and the theorem of Fisher. The extending and generalization of these known results for an arbitrary number *m* of fuzzy metric spaces is obtained using implicit relations introduced as follows: Let Φ_m be the set of continuous functions with *m* variables:

$\varphi:[0,1]^m \to [0,1], m \in N$ with the following properties:

1. φ is no decreasing on $t_1, t_2, \ldots, t_m$ variables and **2.** $\varphi(t,t,\ldots,t) \geq t$ for $\forall t \in [0,1]$.

After that, we prove our theorem from which a several corollaries follow according as the forms of implicit function φ. A counterexample proves that the continuity of $m-2$ between m mappings is necessary.

Key words: Cauchy sequence, fixed point, fuzzy metric space, implicit relation.

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1. INTRODUCTION AND PRELIMINARIES

Fisher [5] and Popa [12] proved some fixed point theorems on two metric spaces. Nung [12] and Jain et al. [8] proved similar results for three metric spaces. Later, using the implicit relation, other authors unified and generalized some of the well-known theorems. So Telci [16] and later Aliouche and Fisher [1] realized the generalization for two mappings on two metric spaces. In this paper, a several known results for two and three metric spaces are generalized and extended in two, three and in general in *m* fuzzy metric spaces.

The concept of fuzzy sets was introduced initially by Zadeh [17]. George and Veeramani [6] modified the concept of fuzzy metric space which was introduced by Kramosil and Michalek [10] and defined a Hausdorff topology in this space. Grabiec [7] extended the well known fixed point theorems of Banach [2] and Edelstein [4] in fuzzy metric spaces. In this paper, using a new class of implicit relations, we prove a theorem as a corollary of which are taken the fuzzyfication of theorems: Nung [12], Jain et al [8], Popa [13], Telci [16], the theorem of Fisher [2] etc.

Firstly, we will give some known definitions and lemmas.

Definition 2.1. [17] A fuzzy set *A* in *X* is a function with domain *X* and values in $[0,1]$.

Definition2.2.[15] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous *t*-norm ,if $([0,1],*)$ is an abelian topological monoid

with the unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.
Two typical examples of continuous *t*-norm are $a*b = ab$ and $a*b = \min(a,b)$.

Definition 2.3.[6] *The 3-tuple* $(X,M,*)$ *is called a fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions:*
For all $x,y,z \in X$ *and* $t,s > 0$,
(FM-1) $M(x,y,t) > 0$,
(FM-2) $M(x,y,t) = 1$ *if and only if* $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$,
(FM-4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$
(FM-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous*.

Example 2.4.[6] Let (X,d) be a metric space. Define $a*b = ab$ and

$$M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}, k,m,n \in R^+.$$

Then $(X,M,*)$ is a fuzzy metric space.
In the above example by taking $k = m = n = 1$ we get $M(x,y,t) = \frac{t}{t + d(x,y)}$.
We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition2.5[7] *Let* $(X,M,*)$ *be a fuzzy metric space. Then*:
(1) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *(denoted by* $\lim_{n\to\infty} x_n = x$*) if* $\lim_{n\to\infty} M(x_n,x,t) = 1$ *for all* $t > 0$.
(2) A sequence $\{x_n\}$ *in* X *is called a Cauchy sequence if* $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ *for all* $t > 0$ *and* $p > 0$.
(3) *A fuzzy metric space in which every Cauchy sequence is convergent is called complete*.

Lemma 2.6.[7] For all $x,y \in X$, $M(x,y,\cdot)$ is no decreasing.

Remark 2.7. Throughout this paper, $(X,M,*)$ will denote the fuzzy metric space in the sense of Definition 2.3 with the following condition:
(FM-6) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$ and $t > 0$.

Lemma 2.8.[14] Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0,\infty)$.

Lemma 2.9. ([16],[17]) Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X,M,*)$ .If there exist a number $k \in (0,1)$ such that $M(y_{n+2},y_{n+1},kt) \ge M(y_{n+1},y_n,t)$ for all $t > 0$ and $n = 1,2,\ldots$then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.10.[17] If for all $x,y \in X$, $t > 0$ and for a number $k \in (0,1)$, $M(x,y,kt) \ge M(x,y,t)$, then $x = y$.

3. IMPLICIT RELATIONS

Let Φ_m be the set of continuous functions with m variables:

$$\varphi : [0,1]^m \to [0,1], m \in N$$

with the following properties:
3.a. φ is no decreasing on $t_1, t_2, \ldots, t_m$ variables and
3.b. $\varphi(t,t,\ldots,t) \ge t$ for $\forall t \in [0,1]$.

We denote $I_m = \{1,2,\ldots,m\}$. The following functions satisfy the above properties:
Example 3.1. $\varphi(t_1,t_2,\ldots,t_m) = \min\{t_1,t_2,\ldots,t_m\}$.
Example 3.2.

$$\varphi(t_1,t_2,\ldots,t_m) = [\min\{t_i t_j : i,j \in I_m\}]^{1/2}.$$

Example 3.3. $\varphi(t_1,t_2,\ldots,t_m) = [\min\{t_1^p, t_2^p, \ldots, t_m^p\}]^{1/p}$.
Example 3.4. $\varphi(t_1,t_2,\ldots,t_m) = t_1 * t_2 * \ldots * t_m$ where $*$ is a *t*-norm such that $t*t \ge t$ as it is the case $a*b = \min\{a,b\}$.
For $m = 5$ we can give these examples:
Example 3.5. $\varphi(t_1,t_2,t_3,t_4,t_5) = t_i$, $i \in I_5$.
Example 3.6. $\varphi(t_1,t_2,t_3,t_4,t_5) = \min\{t_i,t_j\}$, $i,j \in I_5$.

Example 3.7. $\varphi(t_1,t_2,t_3,t_4,t_5)=\min\{t_i,t_j,t_k\}$, $i,j,k\in I_5$.

4. MAIN RESULTS

Firstly, we give the main theorem for *m*=2 and *m*=3 and then we give the theorem for *m* fuzzy metric spaces.

Theorem 4.1 Let $(X,M_1,*_1)$ and $(Y,M_2,*_2)$ be two complete fuzzy metric spaces and $T:X\to Y$, $S:Y\to X$ two maps which satisfy the conditions:

$$M_1(Sy,STx,kt)\geq\varphi_1(M_2(y,Tx,t),M_1(x,Sy,t),M_1(x,STx,t))$$
$$M_2(Tx,TSy,kt)\geq\varphi_2(M_1(x,Sy,t),M_2(y,Tx,t),M_2(y,TSy,t))$$

for all $x\in X$, $y\in Y$, $t>0$ where $k\in(0,1)$ and $\varphi_1,\varphi_2\in\Phi_3$. Then ST has a unique fixed point $\alpha\in X$ and TS has a unique fixed point $\beta\in Y$. Moreover, $T\alpha=\beta$ and $S\beta=\alpha$.

Theorem 4.2 Let $(X,M_1,*_1)$, $(Y,M_2,*_2)$, $(Z,M_3,*_3)$ be three complete fuzzy metric spaces, $T:X\to Y$, $S:Y\to Z$ and $R:Z\to X$ three maps satisfying

$$M_1(RSy,RSTx,kt)\geq\varphi_1(M_2(y,Tx,t),M_3(Sy,STx,t),M_1(x,RSy,t),M_1(x,RSTx,t))\quad(1)$$
$$M_2(TRz,TRSy,kt)\geq\varphi_2(M_3(z,Sy,t),M_1(Rz,RSy,t),M_2(y,TRz,t),M_2(y,TRSy,t))\quad(2)$$
$$M_3(STx,STRz,kt)\geq\varphi_3(M_1(x,Rz,t),M_2(Tx,TRz,t),M_3(z,STx,t),M_3(z,STRz,t))\quad(3)$$

for all $x\in X$, $y\in Y$, $z\in Z$, $t>0$ where $k\in(0,1)$ and $\varphi_1,\varphi_2,\varphi_3\in\Phi_4$. If one of the maps T,S,R is continuous, then RST has a unique fixed point $\alpha\in X$, TRS has a unique fixed point $\beta\in Y$ and STR has a unique fixed point $\gamma\in Z$. Moreover, $T\alpha=\beta$, $S\beta=\gamma$ and $R\gamma=\alpha$.

Proof. Let x_0 be a arbitrary point in X. Construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in X, Y and Z, respectively, as follows:

$$x_n=(RST)^n x_0,\quad y_n=Tx_{n-1},\quad z_n=Sy_n,\ n\in N.$$

We will show that, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Denote:

$$d_n(t)=M_1(x_n,x_{n+1},t)$$
$$\rho_n(t)=M_2(y_n,y_{n+1},t)$$
$$\sigma_n(t)=M_3(z_n,z_{n+1},t)$$

Apply (2) with $z=z_{n-1}$ and $y=y_n$.

$$\rho_n(kt)=M_2(y_n,y_{n+1},kt)=M_2(TRz_{n-1},TRSy_n,kt)$$

Then:
$$\geq\varphi_2(M_3(z_{n-1},z_n,t),M_1(x_{n-1},x_n,t),M_2(y_n,y_n,t),M_2(y_n,y_{n+1},t))\quad(4)$$
$$=\varphi_2(\sigma_{n-1}(t),d_{n-1}(t),1,\rho_n(t))$$

We prove, first, that $\rho_n(t)\geq\min\{\sigma_{n-1}(t),d_{n-1}(t)\}$, $\forall n\in N$.

Suppose that $\rho_n(t)<\min\{\sigma_{n-1}(t),d_{n-1}(t)\}$ for some $n\in N$. Using properties 3.a and 3.b of φ_2, we have

$$\rho_n(kt)\geq\varphi_2(\rho_n(t),\rho_n(t),\rho_n(t),\rho_n(t))\geq\rho_n(t)$$

or

$$M_2(y_n,y_{n+1},kt)\geq M_2(y_n,y_{n+1},t)$$

Now, from Lemma 2.10 it follows that $y_n=y_{n+1}$ and $\rho_n(t)=M_2(y_n,y_{n+1},t)=1$. So, we get $1=\rho_n(t)<\min\{\sigma_{n-1}(t),d_{n-1}(t)\}$. A contradiction! Remember that $\sigma_{n-1}(t),d_{n-1}(t)\in[0,1]$. Hence,

$$\rho_n(t) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \tag{5}$$

Next, from (4) , after the application of 3.a and 3.b,we find

$$\rho_n(kt) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \tag{6}$$

In a similar way, using (3) and (5), we have

$$\begin{aligned} \sigma_n(kt) &= M_3(z_n, z_{n+1}, kt) = M_2(STx_{n-1}, STRz_n, kt) \\ &\geq \varphi_3(M_1(x_{n-1}, x_n, t), M_2(y_n, y_{n+1}, t), M_3(z_n, z_n, t), M_3(z_n, z_{n+1}, t)) \\ &= \varphi_3(d_{n-1}(t), \rho_n(t), 1, \sigma_n(t)) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\} \end{aligned}$$

Thus,

$$\sigma_n(kt) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \tag{7}$$

Analogously, applying (1), we find

$$\begin{aligned} d_n(kt) &= M_1(x_n, x_{n+1}, kt) = M_1(RSy_n, RSTx_n, kt) \\ &\geq \varphi_1(M_2(y_n, y_{n+1}, t), M_1(z_n, z_{n+1}, t), M_1(x_n, x_n, t), M_1(x_n, x_{n+1}, t)) \\ &= \varphi_1(\rho_n(t), \sigma_n(t), 1, d_n(t)) \geq \min\{\rho_n(t), \sigma_n(t)\} \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\} \end{aligned}$$

So,

$$d_n(kt) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\}, \quad \forall n \in N \tag{8}$$

Applying (8) and (7), considering as t the number $\frac{t}{k}$, we obtain

$$d_{n-1}(t) = d_{n-1}(k\frac{t}{k}) \geq \min\{\sigma_{n-2}(\frac{t}{k}), d_{n-2}(\frac{t}{k})\}$$

and

$$\sigma_{n-1}(t) = \sigma_{n-1}(k\frac{t}{k}) \geq \min\{\sigma_{n-2}(\frac{t}{k}), d_{n-2}(\frac{t}{k})\}.$$

By induction we have

$$d_n(kt) \geq \min\{\sigma_{n-1}(t), d_{n-1}(t)\} \geq \min\{\sigma_{n-2}(\frac{t}{k}), d_{n-2}(\frac{t}{k})\} \geq \ldots \geq$$

$$\geq \min\{\sigma_1(\frac{t}{k^{n-2}}), d_1(\frac{t}{k^{n-2}})\}$$

or

$$d_n(t) \geq \min\{\sigma_1(\frac{t}{k^{n-1}}), d_1(\frac{t}{k^{n-1}})\}, \quad \forall n \in N.$$

In the same way

$$\rho_n(t) \geq \min\{\sigma_1(\frac{t}{k^{n-1}}), d_1(\frac{t}{k^{n-1}})\}$$

and

$$\sigma_n(t) \geq \min\{\sigma_1(\frac{t}{k^{n-1}}), d_1(\frac{t}{k^{n-1}})\}.$$

Thus, for $\forall n \in N, t > 0$ we have

$$M_1(x_n, x_{n+1}, t) \geq \min\{M_3(z_1, z_2, \frac{t}{k^{n-1}}), M_1(x_1, x_2, \frac{t}{k^{n-1}})\}$$

$$M_2(y_n, y_{n+1}, t) \geq \min\{M_3(z_1, z_2, \frac{t}{k^{n-1}}), M_1(x_1, x_2, \frac{t}{k^{n-1}})\}$$

$$M_3(z_n,z_{n+1},t)\geq \min\{M_3(z_1,z_2,\frac{t}{k^{n-1}}),M_1(x_1,x_2,\frac{t}{k^{n-1}})\}$$

But $\lim_{n\to\infty}\frac{t}{k^{n-1}}=\infty$ because $k\in(0,1)$ and applying (FM-6) we get

$$\lim_{n\to\infty}\sigma_1(\frac{t}{k^{n-1}})=\lim_{n\to\infty}M_3(z_1,z_2,\frac{t}{k^{n-1}})=1$$

and

$$\lim_{n\to\infty}d_1(\frac{t}{k^{n-1}})=\lim_{n\to\infty}M_1(x_1,x_2,\frac{t}{k^{n-1}})=1$$

Consequently,

$$\lim_{n\to\infty}M_1(x_n,x_{n+1},t)=\lim_{n\to\infty}M_2(y_n,y_{n+1},t)=\lim_{n\to\infty}M_3(z_n,z_{n+1},t)=1\,.$$

Now, for all n and p, we use the Definition 2.3, (FM-4) obtaining

$$M_1(x_n,x_{n+p},t)\geq \underbrace{M_1(x_n,x_{n+1},\frac{t}{p})*M_1(x_{n+1},x_{n+2},\frac{t}{p})*\ldots*M_1(x_{n+p-1},x_{n+p},t)}_{p}$$

When n tends to infinity, we have

$$\lim_{n\to\infty}M_1(x_{n+p},x_n,t)\geq \underbrace{1*1*\ldots*1}_{p}$$

Concluding that, $\lim_{n\to\infty}M_1(x_{n+p},x_n,t)=1$.

This shows that $\{x_n\}$ is a Cauchy sequence in X. We can show in the same way that $\{y_n\}$ and $\{z_n\}$, are also Cauchy sequences in Y and Z, respectively. That is,

$$\lim_{n\to\infty}x_n=\alpha\in X,\ \lim_{n\to\infty}y_n=\beta\in Y,\ \lim_{n\to\infty}z_n=\gamma\in Z\,.$$

Suppose that S is continuous. Then, since $z_n=Sy_n$ taking the limit we have $S\beta=\gamma$ (9)

Applying (1) we get

$$M_1(RS\beta,x_{n+1},kt)=M_1(RS\beta,RSTx_n,kt)\geq$$
$$\geq\varphi_1(M_2(\beta,y_{n+1},t),M_3(S\beta,z_{n+1},t),M_1(x_n,RS\beta,t),M_1(x_n,x_{n+1},t))$$

Now, when n tends to infinity, using (9) we have

$$M_1(RS\beta,\alpha,kt)\geq\varphi_1(1,1,M_1(\alpha,RS\beta,t),1))\geq M_1(\alpha,RS\beta,t)$$

This means (Lemma 2.10) that $RS\beta=\alpha$ (10)

And from (9) we get

$$R\gamma=\alpha \qquad (11)$$

Using (10) and (2), we obtain

$$M_2(T\alpha,y_{n+1},kt)=M_2(TRS\beta,TRSy_n,kt)$$
$$\geq\varphi_2(M_3(S\beta,Sy_n,t),M_1(RS\beta,x_n,t),M_2(y_n,T\alpha,t),M_2(y_n,y_{n+1},t))$$

Letting n tend to infinity we take

$$M_2(T\alpha,\beta,kt)\geq\varphi_2(1,1,M_2(\beta,T\alpha,t),1)\geq M_2(T\alpha,\beta,t)\,.$$

Thus, $T\alpha=\beta$ (12)

Next, from (9),(11) and (12), we have

$$TRS\beta=TR\gamma=T\alpha=\beta\,,$$
$$STR\gamma=ST\alpha=S\beta=\gamma\,,$$

$$RST\alpha = RS\beta = R\gamma = \alpha .$$

So, α is a fixed point for RST, β is a fixed point for TRS and γ is a fixed point for STR.

To prove the uniqueness, we suppose that α' is another fixed point of RST. Applying (1) for $y = T\alpha$ and $x = \alpha'$, we have

$$\begin{aligned} M_1(\alpha,\alpha',kt) &= M_1(RST\alpha, RST\alpha', kt) \geq \\ &\geq \varphi_1(M_2(T\alpha,T\alpha',t), M_3(ST\alpha,ST\alpha',t), M_1(\alpha', RST\alpha,t), M_1(\alpha', RST\alpha',t)) \\ &= \varphi_1(M_2(T\alpha,T\alpha',t), M_3(ST\alpha,ST\alpha',t), M_1(\alpha,\alpha',t),1) \end{aligned}$$

Applying now 3.a and 3.b for φ_1, obtain

$$M_1(\alpha,\alpha',kt) \geq \min\{M_2(T\alpha,T\alpha',t), M_3(ST\alpha,ST\alpha',t)\} \qquad (13)$$

Next, from (2) it follows that

$$\begin{aligned} M_2(T\alpha,T\alpha',kt) &= M_2(TRST\alpha, TRST\alpha',kt) \geq \\ &\geq \varphi_2(M_3(ST\alpha,ST\alpha',t), M_1(RST\alpha, RST\alpha',t), M_2(T\alpha', TRST\alpha,t), M_2(T\alpha',T\alpha',t)) \geq \\ &= \varphi_2(M_3(ST\alpha,ST\alpha',t), M_1(\alpha,\alpha',t), M_2(T\alpha',T\alpha,t),1) \end{aligned}$$

Thus, we have, $M_2(T\alpha,T\alpha',kt) \geq \min\{M_3(ST\alpha,ST\alpha',t), M_1(\alpha,\alpha',t)\}$ (14)

Now, from (13) and (14) and from the fact that $M_2(T\alpha,T\alpha',t) \geq M_2(T\alpha,T\alpha',kt)$, we have

$$M_1(\alpha,\alpha',kt) \geq M_3(ST\alpha,ST\alpha',t) \qquad (15)$$

Finally, from (3), it follows that

$$\begin{aligned} M_3(ST\alpha,ST\alpha',kt) &= M_3(STRST\alpha, STRST\alpha',kt) \geq \\ &\geq \varphi_3(M_1(\alpha,\alpha',t), M_2(T\alpha,T\alpha',t), M_3(ST\alpha',ST\alpha,t), M_3(ST\alpha',ST\alpha',t)) \\ &= \varphi_3(M_1(\alpha,\alpha',t), M_2(T\alpha,T\alpha',t), M_3(ST\alpha',ST\alpha,t),1) \end{aligned}$$

Hence, $M_3(ST\alpha,ST\alpha',kt) \geq \min\{M_1(\alpha,\alpha',t), M_2(T\alpha,T\alpha',t)\}$ (16)

Again, from (14), (15) and (16) and from the fact that $M_3(ST\alpha,ST\alpha',t) \geq M_3(ST\alpha,ST\alpha',kt)$ we have

$$\begin{aligned} M_1(\alpha,\alpha',kt) &\geq M_3(ST\alpha,ST\alpha',t) \geq M_3(ST\alpha,ST\alpha',kt) \geq \\ &\geq \min\{M_1(\alpha,\alpha',t), M_2(T\alpha,T\alpha',t)\} = M_2(T\alpha,T\alpha',t) \geq \\ &\geq M_2(T\alpha,T\alpha',kt) \geq \min\{M_3(ST\alpha,ST\alpha',t), M_1(\alpha,\alpha',t)\} = \\ &= M_3(ST\alpha,ST\alpha',t) \end{aligned}$$

From the inequalities $M_1(\alpha,\alpha',kt) \geq M_3(ST\alpha,ST\alpha',kt) \geq M_3(ST\alpha,ST\alpha',t)$ it follows that

$$ST\alpha = ST\alpha',\ M_3(ST\alpha,ST\alpha',kt) = 1 \text{ and } M_1(\alpha,\alpha',kt) \geq 1 .$$

So, $\alpha = \alpha'$.

Thus, α is the unique fixed point for RST. In the same way we show that β is the unique fixed point for TRS and γ the unique fixed point for STR. This completes the proof.

Theorem 4.3 Let $(X_i, M_i, \bullet_i)$ be m complete metric spaces and let T_i m mappings such that $T_i : X_i \to X_{i+1}$ for $i = 1,2,\ldots,m-1$, $T_m : X_m \to X_1$ and from which $(m-2)$ are continuous. If for some $c \in (0,1)$ and $\varphi_i \in \Phi_{m+1}$ the inequalities are satisfied:

$$M_1(T_m T_{m-1} \ldots T_2 x_2, T_m T_{m-1} \ldots T_2 T_1 x_1, ct) \geq$$
$$\geq \varphi_1 \begin{pmatrix} (M_1(x_1, T_m T_{m-1} \ldots T_2 x_2, t), M_1(x_1, T_m T_{m-1} \ldots T_1 x_1, t), M_2(x_2, T_1 x_1, t), \\ M_3(T_2 x_2, T_2 T_1 x_1, t), \ldots, M_m(T_{m-1} T_{m-2} \ldots T_3 T_2 x_2, T_{m-1} T_{m-2} \ldots T_2 T_1 x_1, t) \end{pmatrix} \qquad (1)$$

for all $x_1 \in X_1$ and $x_2 \in X_2$

$$M_2\left(T_1T_m\ldots T_4T_3x_3,\,T_1T_m\ldots T_3T_2x_2,\,ct\right)\geq$$
$$\varphi_2\begin{pmatrix}M_2\left(x_2,\,T_1T_mT_{m-1}\ldots T_3x_3,\,t\right),M_2\left(x_2,\,T_1T_mT_{m-1}\ldots T_2x_2,\,t\right),M_3\left(x_3,\,T_2x_2,\,t\right),\\ M_4\left(T_3x_3,\,T_3T_2x_2,\,t\right),\ldots,M_m\left(T_{m-1}T_{m-2}\ldots T_3x_3,\,T_{m-1}T_{m-2}\ldots T_2x_2,\,t\right),\\ M_1\left(T_mT_{m-1}\ldots T_3x_3,\,T_mT_{m-1}\ldots T_2x_2,\,t\right)\end{pmatrix} \quad (2)$$

for all $x_2\in X_2$ and $x_3\in X_3$, in general

$$M_i\left(T_{i-1}T_{i-2}\ldots T_1T_mT_{m-1}\ldots T_{i+1}x_{i+1},T_{i-1}T_{i-2}\ldots T_1T_mT_{m-1}\ldots T_ix_i,ct\right)\geq$$
$$\varphi_i\begin{pmatrix}M_i\left(x_i,T_{i-1}T_{i-2}\ldots T_1T_mT_{m-1}\ldots T_{i+1}x_{i+1},t\right),M_i\left(x_i,T_{i-1}T_{i-2}\ldots T_1T_mT_{m-1}\ldots T_ix_i,t\right),M_{i+1}\left(x_{i+1},T_ix_i,t\right),\\ M_{i+2}\left(T_{i+1}x_{i+1},T_{i+1}T_ix_i,t\right),\ldots,M_m\left(T_{m-1}T_{m-2}\ldots T_{i+1}x_{i+1},T_{m-1}T_{m-2}\ldots T_ix_i,t\right),\\ M_1\left(T_mT_{m-1}\ldots T_{i+1}x_{i+1},T_mT_{m-1}\ldots T_ix_i,t\right),M_2\left(T_1T_mT_{m-1}\ldots T_{i+1}x_{i+1},T_1T_mT_{m-1}\ldots T_ix_i,t\right),\ldots,\\ M_{i-1}\left(T_{i-2}T_{i-3}\ldots T_1T_mT_{m-1}\ldots T_{i+1}x_{i+1},T_{i-2}T_{i-3}\ldots T_1T_mT_{m-1}\ldots T_ix_i,t\right)\end{pmatrix} \quad (i)$$

for all $x_i\in X_i$, $x_{i+1}\in X_{i+1}$ for $i=3,\ldots,m-1$, and

$$M_m\left(T_{m-1}T_{m-2}\ldots T_1x_1,\,T_{m-1}T_{m-2}\ldots T_1T_mx_m,\,ct\right)\geq$$
$$\varphi_m\begin{pmatrix}M_m\left(x_m,T_{m-1}T_{m-2}\ldots T_1x_1,t\right),M_m\left(x_m,T_{m-1}T_{m-2}\ldots T_1T_mx_m,t\right),M_1\left(x_1,T_mx_m,t\right),\\ M_2\left(T_1x_1,T_1T_mx_m,t\right),\ldots,M_{m-1}\left(T_{m-2}T_{m-3}\ldots T_1x_1,T_{m-2}T_{m-3}\ldots T_1T_mx_m,t\right)\end{pmatrix} \quad (m)$$

for all $x_1\in X_1$ and $x_m\in X_m$, where $\varphi_i\in\Phi_{m+3}$ for $i=1,2,\ldots,m$. Then the maps $T_mT_{m-1}\ldots T_2T_1$, $T_1T_mT_{m-1}\ldots T_2$, ..., $T_{i-1}T_{i-2}\ldots T_1T_mT_{m-1}\ldots T_i$,..., $T_{m-1}T_{m-2}\ldots T_1T_m$ have unique fixed point $\alpha_1\in X_1$, $\alpha_2\in X_2$, ..., $\alpha_i\in X_i$, ..., $\alpha_m\in X_m$, respectively. Further,
$T_i\alpha_i=\alpha_{i+1}$ for $i=1,\ldots,m-1$ and $T_m\alpha_m=\alpha_1$.
This theorem is proved in the same way as the theorem 4.2.

5. COROLLARIES

Corollary 5.1 From Theorem 4.3 for $m=2$ we take the Theorem 4.1 which generalizes and fuzzyfies the Theorems Fisher [5], Popa [13] etc.

Corollary 5.2 If in Corollary 5.1 (theorem 4.1) we take $\varphi_1=\varphi_2=\varphi\in\Phi_3$ where $\varphi(t_1,t_2,t_3)=\min\{t_1,t_2,t_3\}$ we obtain the theorem which fuzzyfies the Fisher theorem. (Theorem 1.[5]) for metric spaces.

Corollary 5.3 If in Corollary 5.1 (theorem 4.1) we take $\varphi_1=\varphi_2=\varphi\in\Phi_3$ where $\varphi(t_1,t_2,t_3)=[\min\{t_1t_2,t_1t_3,t_2t_3\}]^{1/2}$, we obtain the fuzzyfication of Popa result (Theorem 1[13]) for metric spaces.

Corollary 5.4 If in Corollary 5.1 (theorem 4.1) we take $\varphi_1=\varphi_2=\varphi\in\Phi_3$ where $\varphi(t_1,t_2,t_3)=[\min\{t_1^p,t_2^p,t_3^p\}]^{1/p}$, $p>0$, we obtain a generalization of Corollary 5.2 which is taken for $p=1$.

Remark 5.5 We can obtain many other similar results for different φ.

Corollary 5.6 From the Theorem 4.3 for $m=3$ we take the Theorem 4.2 which generalizes and fuzzyfies the Theorems Nung [12], Jain et al [8], Kikina[9], etc.

Corollary 5.7 If in theorem 4.3 we take $\varphi=\varphi_1=\varphi_2=\varphi_3\in\Phi_4$ where $\varphi(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}$, we obtain the theorem which fuzzyfies the result of Nung [12] for metric spaces.

Corollary 5.8 If in theorem 4.3 we take $\varphi=\varphi_1=\varphi_2=\varphi_3\in\Phi_4$ where

$\varphi(t_1,t_2,t_3,t_4)=[\min\{t_1t_3,t_1t_4,t_2t_3,t_2t_4\}]^{1/2}$, we take the theorem of Jain, Shrivastava and Fisher (Theorem 2 [8]).

Corollary 5.9 If

$\varphi(t_1,t_2,t_3,t_4)=[\min\{t_1^p,t_3^p,t_4^p\}]^{1/p}$, we take the result of Kikina (Theorem 2.1[9], $F=0$) for the metric spaces and for

$\varphi(t_1,t_2,t_3,t_4)=[\min\{t_1^p,t_2^p,t_3^p,t_4^p\}]^{1/p}$ we take its generalization.

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit functions.

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