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FIXED POINTS FOR WEAKLY CONTRACTIVE MAPPINGS IN b-METRIC SPACES
PIKA FIKSE TE FUNKSIONEVE DOBESISHT KONTRAKTIVE NE HAPESIRAT b-METRIKE

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PËRMBLEDHJE
b-distanca përbën një përgjithësim të distancës klasike, në të cilën kemi një kusht më të përgjithshëm se ai i mosbarazimit të trekëndëshit: $d(x,y) \leq s(d(x,z)+d(z,y))$ ku $s \geq 1$. Si motivim për hyrjen e konceptit të b-distancës ka shërbyer nevoja për të patur një model matematik të përshtatshëm ne teorinë e llogaritjeve dhe në studimin e konvergjencës sipas masës të funksioneve të matshëm. Qëllimi i këtij punimi është të përkufizojë klasën e funksioneve dobësisht kontraktive në hapësirat b-metrike dhe të studjojë kondita të mjaftueshme të ekzistencës dhe unicitetit të pikave fikse te ketyre funksioneve në kuadrin e hapësirave të plota b-metrike. Gjithashtu ne mbeshtetje te reyultateve tona japim edhe disa shembuj që mbeshtetes. Konkluzione: në këtë punim ne iniciojmë fillimin e studimit te funksioneve dobësisht kontraktive në hapësirat b-metrike dhe rezultatet tona ndihmojnë studjuesit e këtyre hapësirave në zgjidhjen e problemeve me ndihmën e pikave fikse.
Fjalët çelës: b-metric space, weakly contractive condition.

SUMMARY
The b-metric is a generalizations of classical metric, which satisfies a relaxed triangle inequality $d(x,y) \leq s(d(x,z)+d(z,y))$ for some constant $s \geq 1$, rather than the usual triangle inequality. A motivation behind introducing the concept of b-metric was to obtain appropriate mathematical models in the theory of computation, in the study of the problem of the convergence of measurable functions with respects to measure. The aim of this work is to define the weakly contractive mappings in b-metric space and using the methods of successive approximations to study the necessary conditions for existence and uniqueness of fixed points of them in the framework of complete b-metric spaces. Also we provide some examples to validate our results. Conclusions; our results initiate the study of weakly contractive mappings in complete b-metric and help to solve numerous problems through fixed point theory.
Key-words: b-metric space, weakly contractive condition.

INTRODUCTION

The concept of weakly contractive point-to-point mappings is introduced by Alber and Guerr-Delabriere [1] in the settings of Hilbert spaces. Rhoades [11] showed that most results of [1] are still true for any Banach space. Also Bae [4] obtain fixed point theorems of multivalued weakly contractive mapping. Zhang and Song [14] proved a common fixed point theorem for a pair of generalized φ-weak contractions in complete metric space.

The b-metric first introduced by Czerwik [9] as a generalization of classical metric. There exists a numerous generalizations of the Banac'h contraction Principle in literature in single-valued and multi-valued operators in b-metric spaces [2, 3, 5, 6, 7, 8, 12, 13]. On the other hand in recent years, many authors has been considered the weakly contractive mappings in generalized metric spaces. In this work we initiate the study of fixed point theory for weakly contractive mappings in the settings of complete b-metric spaces.

PRELIMINARIES.

Definition 2.1[9] Let X be a set and let $s \geq 1$ be given real number. A function $d: X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$;
3. $d(x,z) \leq s[d(x,y)+d(y,z)]$

The pair (X,d) is called a b-metric space with parameter s.

There exists more examples in the literature [5, 7] showing that the class of b-metrics is effectively larger than that of metric spaces, since a b-metric is a metric when $s=1$ in the above condition 3.

Example. The space $l_p (0<p<1)$,

$$l_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

together with the function $d: l_p \times l_p \to \mathbb{R}$,

$$d(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

where $x=(x_n), y=(y_n) \in l_p$ is a b-metric space with parameter $s = 2^{\frac{1}{p}} > 1$.

Example. The space $L_p (0<p<1)$ of all real functions $x(t), t \in [0,1]$ such that:

$$\int_o^1 |x(t)|^p dt < \infty$$

is a b-metric space if we take

$$d(x,y) = (\int_0^1 |x(t)-y(t)|^p dt)^{\frac{1}{p}} \text{ for each } x,y \in L_p.$$

The parameter $s = 2^{\frac{1}{p}} > 1$

As in the metric case, the b-metric d induces the topology $\tau(d) = \{G \subseteq X : \forall x \in G, \exists r>0 : B(x,r) \subseteq G\}$, where $B(x,r) = \{y \in X : d(x,y) < r\}$ consists the open ball in the b-metric space (X,d). The notions of convergence, compactness, closedness and completeness in b-metric spaces given in the some way as in metric spaces.

In general, a *b*-metric is not continuous.

Lemma 2.2 [12] Let (X,d) be a b-metric space with parameter s and $(x_n)_{n \in \mathbb{N}}$ a sequence in X such that:

$$d(x_{n+1}, x_{n+2}) \leq q d(x_n, x_{n+1}),\ n \in \mathbb{N}$$

where $0 \leq q < 1$. Then the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy sequence in X provided that $sq < 1$.

Definition 2.3 [8] Let (X,d) be a b-metric space with parameter s. The Hausdorff b-metric H on $C(X)$; the collection of all nonempty compact subsets of (X,d) is defined as follows:

$$H(A,B) = \begin{cases} \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\} & \text{if it exists} \\ \infty, & \text{otherwise} \end{cases}$$

It is known [8] that $(C(X),H)$ is a complete metric space provided (X,d) is a complete metric space. We cite the following Lemma:

Lemma 2.4 [8] Let (X,d) be a b-metric space with parameter s and $A,B,C \in C(X)$.
Then:
(i) $d(x,B) \leq d(x,y)$ for any $y \in B$,
(ii) $d(A,B) \leq H(A,B)$
(iii) $d(x,B) \leq H(A,B)$, $x \in A$,

(iv) $H(A,C) \le s[H(A,B)+H(B,C)]$
(v) $d(x,A) \le s[d(x,y)+d(y,A)],\ x,y \in X$.

Let (X,d) be a b-metric space with parameter s .

Definition 2.5 A mapping $T: X \to X$ is said to be a weakly contractive if for all $x,y \in X$

$$d(T(x),T(y)) \le d(x,y) - \varphi(d(x,y)) \qquad (1)$$

where $\varphi:[0,+\infty) \to [0,+\infty)$ is a continuous and non-decreasing function such that $\varphi(0)=0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.6 [7] Let $s \ge 1$ be a real number. A mapping $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is called a b-comparison function if :

i) φ is monotone increasing;
ii) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$s^{k+1}\varphi^{k+1}(t) \le a s^k \varphi^k(t) + v_k$$

for $k > k_0$ and any $t \in \mathbb{R}^+$.

Lemma 2.7 [5] If $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is a b-comparison function, then:

1) the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in \mathbb{R}^+$
2) the function $s_b(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t),\ t \in \mathbb{R}^+$, is increasing and continuous at 0.
3) φ is continuous at 0 and $\varphi(0)=0$.

MAIN RESULTS

Theorem 3.1 Let (X,d) be a complete b-metric space with parameter s and with continuous b-metric, $T: X \to X$ be a weakly contractive mapping with φ according to Definition 2.6.

If one of the following two conditions is satisfied:

(a) T is continuous self map on X ;
(b) for any non-decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) with $\lim_{n\to\infty} x_n = z$ it follows $x_n \le z$ for all $n \in \mathbb{N}$.
Then T has a unique fixed point.

Proof. (a) Let $x_0 \in X$ and $x_n = T(x_{n-1}), n \ge 1$.
Then

$$\begin{aligned} d(x_1,x_2) &= d(T(x_0),T(x_1)) \\ &\le d(x_0,x_1) - \varphi[d(x_0,x_1)] < d(x_0,x_1) \end{aligned} \qquad (2)$$

Thus we construct the following sequence $(x_n)_{n\in\mathbb{N}}$ such that:

$$\begin{aligned} & d(x_n,x_{n+1}) \le \\ & d(x_{n-1},x_n) - \varphi(d(x_{n-1},x_n)) < d(x_{n-1},x_n) \end{aligned} \qquad (3)$$

Put $d_n = d(x_n,x_{n+1})$. Then we have

$$d_n \le d_{n-1} - \varphi(d_{n-1}) < d_{n-1}$$

Therefore $(d_n)_{n\in\mathbb{N}}$ is nonnegative non-increasing sequence and hence possesses a limit d^* .
From (3), taking limit when $n \to \infty$, we get:

$$d^* \le d^* - \varphi(d^*) \le d^*$$

And by our assumptions about φ , $d^* = 0$.

In what follows we will show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

As d is a b-metric we obtain:

$$\begin{aligned} & d(x_n,x_{n+p}) \le \\ & \le s d(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots + s^p d(x_{n+p-1},x_{n+p}) \\ & \le s d(x_0,x_1) + s^2 d(x_0,x_1) + \ldots + s^p d(x_0,x_1) \\ & \le \frac{1}{s^{n-1}} [s^n d(x_0,x_1) + s^{n+1} d(x_0,x_1) + \ldots + s^{n+p-1} d(x_0,x_1)] \end{aligned}$$

Denoting $S_n = \sum_{k=0}^{n} s^k d(x_0,x_1),\ n \ge 1$ we have:

$$d(x_n, x_{n+p}) \le \frac{1}{s^{n-1}}(S_{n+p-1} - S_{n-1}), n \ge 1, p \ge 1.$$

Since the series $\sum_{k=0}^{\infty} s^k d(x_0, x_1)$ converges, so there is $S = \lim_{n\to\infty} S_n$ and we obtain that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete b-metric space (X,d) and so there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

The continuity of T and d implies that z is fixed point.

(b) In what follows that the Theorem is still valid for T not necessarily continuous.

Following the proof in a) we only have to check that $T(z) = z$.

In fact,

$$\begin{aligned} d(T(z), z) &\le s(d(T(z), T(x_n) + d(T(x_n), z)) \\ &\le s(d(z, x_n) - \varphi(d(z, x_n)) + d(x_{n+1}, z)) \end{aligned}$$

And taking limit as $n \to \infty$, $d(T(z), z) \le 0$ and this prove that $d(T(z), z) = 0$. Consequently $T(z) = z$.

The uniqueness of fixed point follows from (1).

Example 3.2 Let $X = [0,1]$ be endowed with the b-metric d defined for all $x, y \in X$ by $d(x,y) = |x-y|^2$. It is easy to see that d is a b-metric with parameter $s = 2$ and (X,d) is complete. Also d is not a metric on X.

Define $T: X \to X$ by $T(x) = x - \frac{x^2}{2}$ and $\varphi(t) = \frac{t^2}{2}$.

Then,

$$\begin{aligned} d(T(x), T(y)) &= \left|(x - \frac{x^2}{2}) - (y - \frac{y^2}{2})\right|^2 \\ &= \left|(x-y) - (\frac{x^2}{2} - \frac{y^2}{2})\right|^2 \\ &\le |x-y|^2 - \frac{1}{2}|x-y|^4 \\ &= d(x,y) - \varphi(d(x,y)) \end{aligned}$$

By Theorem 3.1 $x = 0$ is the unique fixed point of T.

Example 3.3 Let $X = [0, +\infty)$ be endowed with the b-metric d defined for all $x, y \in X$ by $d(x,y) = |x-y|^2$. It is easy to see that d is a b-metric with parameter $s = 2$ and (X,d) is complete. Also d is not a metric on X.

Let $T: X \to X$ defined by $T(x) = \frac{x}{5}$ and $\varphi(t) = \frac{3}{4}t$ for all $t \ge 0$.

Then

$$\begin{aligned} d(T(x), T(y)) &= \left|\frac{x}{5} - \frac{y}{5}\right|^2 = \frac{|x-y|^2}{25} \le |x-y|^2 - \frac{3}{4}|x-y|^2 \\ &= d(x,y) - \varphi(d(x,y)) \end{aligned}$$

By Theorem 3.1 $x = 0$ is the unique fixed point of T.

Remark 3.4 If we take in Theorem 3.1 $\varphi(t) = (1-k)t$ with $k < \frac{1}{s^2}$ we obtain the Banach Contraction Principle in settings of complete b-metric spaces.

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